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Evaluation of higher-order theories of piezoelectric plates in bending and in stretching

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Abstract

Many models for the flexural and membranal behaviour of piezoelectric plates are available in the literature. They are based on different assumptions concerning the strain, stress, electric and electric-displacement fields inside the plate. A critical comparison among such models is presented here in a completely analytic way, in order to assess the accuracy of the results they provide and determine their range of applicability. The comparison is made by using a class of case-study problems, whose analytical solutions in the framework of the linear theory of piezoelectricity are available, as benchmarks for the solutions supplied by the plate models. The evaluated models are also here rationally derived from the three-dimensional theory of piezoelectricity, and a consistent treatment of the stress and electric-displacement relaxation conditions is proposed. © 2001 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Intelligent technologies employed in civil, aeronautic and space structures are mainly based on the use of sensors and actuators, frequently made of adaptive materials. Many kinds of smart materials are currently available: among others, we recall here piezoelectric materials, shape memory alloys, magnetostrictive materials, electro-rheological fluids and so on. In particular, piezoelectric materials exhibit a coupling between mechanical and electric fields, and are able to convert mechanical energy into electric energy and vice versa. This ability can be exploited for manufacturing sensors and actuators that, due to their smallness and lightness, can be easily embedded into a structure in order to build up an intelligent assembly. As an example, a piezoelectric laminate is made of different layers, some of which are comprised by a piezoelectric material: these layers are either surface bonded or embedded in the host laminate. The use of such electroded piezoelectric layers as sensors and/or actuators has been having an increasing role in smart structures technologies (e.g., Crawley and deLuis, 1987; Chee et al., 1998). In fact, through the piezoelectric coupling owned by the piezoelectric layers, it is possible to sense and to actuate the deformations of the laminate.

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Therefore, piezoelectric layers turn out to be very effective in noise, vibrations, flutter and shape control of structures. Of course, satisfactory control strategies must be based on reliable modelizations of the piezoelectric laminate. Single-layer modelizations, assuming that one analytical representation of each unknown mechanical and electric field prevails in the whole thickness of the laminate, are hardly able to take into account appropriately the complex mechanical and electric behaviour of the laminate. Hence, the use of multi-layer (or layerwise) modelizations of piezoelectric laminates is recommended. Layerwise modelizations are built up on the basis of models for the membranal and flexural behaviour of each layer (Bisegna et al., 1999). As a consequence, the modelizations of the membranal and flexural behaviour of a homogeneous piezoelectric plate play a central role in layerwise theories of piezoelectric laminates, and hence in smart structures technologies. This consideration motivates the present work, aimed to a classification and comparison of piezoelectric plate models.

A review article summarizing the development of higher-order theories for the analysis of piezoelectric plates was recently published by Wang and Yang (2000). Most of theories discussed there are based on expansions of the mechanical displacement and electric potential as sums of terms, each being the product between an unknown function of the in-plane variables and a known base function of the thickness variable. The energy functional governing each theory is usually obtained by bringing the assumed expansions into some variational formulation of the linear theory of piezoelectricity and integrating over the thickness. Then, the governing two-dimensional equations for the unknown functions of the in-plane variables, along with appropriate boundary conditions, are easily obtained as stationary conditions for the relevant energy functional.

Depending on the different base functions adopted, Wang and Yang (2000) classified existing theories as theories based on power series expansions and theories based on trigonometric expansions. The former ones, considered here, are the simplest and most widely used. They trace back to a paper by Mindlin (1952), where a generalization of Reissner (1945) and Mindlin (1951) pioneering works on elastic plates was presented. Then, Tiersten and Mindlin (1962) used power series expansions for the mechanical displacement and electric displacement fields in order to obtain two-dimensional equations of piezoelectric plates, also including extensional modes. Mindlin (1972, 1984) used full power series expansions of the displacement and electric potential fields and obtained equations for coupled flexural, thickness-shear, extensional and thickness-stretch motions. Among other important contributions, the papers by Lee, Yong, Dökmeci and Tzou should be mentioned. For a comprehensive reference list the reader is referred to Wang and Yang (2000).

When dealing with power series expansions, the main point is to know how many terms are to be taken into account in order to have a sufficiently accurate description of the mechanical behaviour of the plate, or, equivalently, how to truncate the series properly at the end (Wang and Yang, 2000). This paper is mainly addressed to focus on this point.

Indeed, instead of power series expansions, a completely equivalent approach based on expansions of orthogonal Legendre polynomials is considered here (Mindlin and Medick, 1959), in order to make some coupling coefficients between different order terms of the expansions disappear and have simpler expressions of the governing energy functionals. Moreover, the representations of the displacement and electric potential fields are here obtained as a consequence of hypotheses on the transversal elongation (i.e., thickness-stretch), transversal shear (i.e., thickness-shear) and transversal electric fields.

Of course, once a set of plate equations has been obtained, the accuracy of the results it provides has to be tested and its range of applicability has to be determined. This could be done comparing the dispersion relations of simple wave solutions to the plate equations with the corresponding exact relations predicted by the three-dimensional equations. Such a procedure, however, allows to perform only a global evaluation of plate theories, since it gives no information about the pointwise values of the estimated mechanical and electric unknown fields, particularly useful in some applications (e.g., fatigue failure or delamination problems).

In this paper, a different evaluation procedure is adopted. A special class of problems is considered, consisting of a simply supported square piezoelectric plate, grounded along its lateral boundary and loaded by different static sinusoidal mechanical or electric loads. The plate is comprised by transversely isotropic piezoelectric material, with poling direction orthogonal to the plate surfaces. Hence, the flexure and shear-thickness behaviour is uncoupled from the extensional and stretch-thickness behaviour. The displacement, electric-potential, stress and electric-displacement analytical solutions supplied by piezoelectric plate models are compared with the solutions obtained in the framework of the three-dimensional Voigt theory of piezoelectricity (Bisegna and Maceri, 1996a).

The comparison is performed in a completely analytic way. In this manner, the lacking of any term in the estimated solutions clearly emphasizes an intrinsic deficiency of the evaluated theory.

In addition, the comparison is performed for very thin plates (more precisely, in the limit of plate thickness aspect ratio approaching zero), since in this case closed-form analytic exact three-dimensional solutions are available (Bisegna and Maceri, 1996a). A stronger argument to investigate the performance of piezoelectric plate theories for very thin plates rests on the observation that a thick plate (requiring a very-high-order theory) could be conveniently medelled as a stack of several thin layers (requiring intermediate-order theories). The main advantage in doing so is the reduction of the fast oscillations in the thickness that could be exhibited by very-high-order theories. The optimal choice of the intermediate-order theory to be adopted to model each layer is an interesting question, addressed here. Of course, such a theory is the lowest-order one able to exactly reproduce all the unknown fields supplied by the three-dimensional solutions, in the limit of plate thickness aspect ratio approaching zero.

The second major point addressed in this paper concerns the so-called "relaxation conditions", to be introduced when some kinematical term (e.g., the thickness-stretch deformation accompanying extension or flexure due to Poisson's effect) is not taken into account into the chosen expansion. A thorough and rational use of relaxation conditions is performed here. In particular, the relaxation condition on the stress field is chosen to exactly correspond to the hypothesis on the strain field implied by the assumed representation of the displacement field. As an example, in the models based on the Lo et al. (1977, 1978) kinematics, the transversal elongation is linear in the thickness coordinate and the transversal shear is quadratic, and hence the second derivative of the transversal normal stress with respect to the thickness coordinate and the third derivative of the transversal shear stress are assumed here to vanish. Moreover, the same treatment is reserved to the relaxation condition on the electric displacement. As an example, in models assuming a potential varying quadratically in the thickness coordinate, the transversal electric field is linear in the thickness coordinate, and the second derivative of the transversal electric displacement is assumed to vanish.

A recent method (Bisegna and Sacco, 1997; Bisegna, 1997) is used to treat relaxation conditions. Both the a priori assumptions concerning the strain and electric fields, and the relaxation conditions on the stress and electric-displacement fields, on which each plate theory relies, are regarded here as internal constraints (Podio-Guidugli, 1989). As a consequence, plate theories are arranged into the framework of the constrained three-dimensional piezoelectricity. The imposed constraints are assumed to be frictionless: hence, the powerful and general Lagrange multipliers theory can be used. Herein the variational principles governing the constrained problems are explicitly derived.

The paper is organized as follows: In Section 2 the notations are defined and the piezoelectric plate problem is posed. It is approached via a variational formulation, given in Section 3, especially useful in order to derive the constrained problem. Sections 4 and 5 are respectively devoted to deduce some higher-order models for the bending and stretching of piezoelectric plates in the framework of the constrained piezoelectricity. For each model, the underlying hypotheses, the energy functional and the field equations are explicitly reported. For the sake of brevity, the compatible boundary conditions are omitted, but they can be straightforwardly obtained from the stationary conditions for the relevant functionals. The class of problems used in the comparison of the plate models is presented in Section 6. Finally, the results of the comparison are reported in Section 7.

2. Notation and problem position

A Cartesian frame (O, x_1, x_2, x_3) is introduced. Let Ω be a regular region in the (x_1, x_2) plane. A plate with thickness h and middle cross-section Ω is considered: its reference configuration is the domain $\mathscr{P} = \Omega \times (-h/2, h/2)$. The plate is comprised by homogeneous, transversely isotropic, linearly piezoelectric material, with transverse-isotropy axis parallel to x_3. It is well known (Ikeda, 1990) that the constitutive behaviour of such a material can be completely described by five closed-circuit elastic constants (here, as usual, denoted by c_{11}, c_{12}, c_{13}, c_{33} and c_{44}), two clamped permittivity constants (denoted by ε_{11} and ε_{33}) and three piezoelectric constants (denoted by e_{31}, e_{33} and e_{15}). Moreover, the following material constants are introduced for a later use:

$$\begin{aligned}
&\bar{\varepsilon}_{33} = \varepsilon_{33} + e_{33}^2/c_{33}, && \hat{\varepsilon}_{33} = (c_{13}\varepsilon_{33} + e_{31}e_{33})/c_{33} \\
&\tilde{\varepsilon}_{33} = (\varepsilon_{33}c_{11} + e_{31}^2)/c_{33}, && \bar{e}_{31} = e_{31} - e_{33}c_{13}/c_{33} \\
&\hat{e}_{33} = (e_{33}c_{11} - e_{31}c_{13})/c_{33}, && \bar{\varepsilon}_{11} = \varepsilon_{11} + e_{15}^2/c_{44} \\
&\bar{c}_{11} = c_{11} - c_{13}^2/c_{33}, && \bar{c}_{12} = c_{12} - c_{13}^2/c_{33} \\
&\hat{c}_{11} = \bar{c}_{11} + \bar{e}_{31}^2/\bar{\varepsilon}_{33}, && \hat{c}_{12} = \bar{c}_{12} + \bar{e}_{31}^2/\bar{\varepsilon}_{33} \\
&c_{66} = (c_{11} - c_{12})/2, && \nu = c_{12}/c_{11} \\
&\bar{\nu} = \bar{c}_{12}/\bar{c}_{11}, && \hat{\nu} = \hat{c}_{12}/\hat{c}_{11}
\end{aligned} \tag{1}$$

The plate is acted upon by volume forces ${}^3\mathbf{b} = (b_1, b_2, b_3)$, volume charges μ, surface forces ${}^3\mathbf{p}^{\pm} = (p_1^{\pm}, p_2^{\pm}, p_3^{\pm})$ on the upper $(+)$ and lower $(-)$ face, respectively, and surface charges $\omega^{\pm}$. Without loss of generality, it can be supposed that these loads induce either a purely flexural or membranal behaviour, since it is well known that any other loading condition can be obtained by superposition of such loads.

Of course, the loads inducing a purely flexural behaviour have the following properties: b_1 and b_2 are odd functions of the x_3 variable, whereas b_3 and μ are even functions of x_3; moreover, $p_1^+ = -p_1^-$, $p_2^+ = -p_2^-$, $p_3^+ = p_3^-$ and $\omega^+ = \omega^-$. On the other hand, the loads inducing a purely membranal behaviour are such that b_1 and b_2 are even functions of the x_3 variable, b_3 and μ are odd functions of x_3, and $p_1^+ = p_1^-$, $p_2^+ = p_2^-$, $p_3^+ = -p_3^-$ and $\omega^+ = -\omega^-$.

For the sake of simplicity and brevity, as far as the theoretical derivation of the plate models is concerned, the plate is assumed to be clamped and grounded along its lateral boundary $\partial\Omega \times (-h/2, h/2)$. On the other hand, in the analysis of the case-study problems, a simply supported plate is considered.

In the framework of the linear theory of piezoelectricity (Tiersten, 1969), the unknowns of the piezoelectric problem for $\mathscr{P}$ are the displacement field ${}^3\mathbf{s}$, the strain field ${}^3\mathbf{E}$, the stress field ${}^3\mathbf{T}$, the electric-potential field ϕ, the electric field ${}^3\mathbf{e}$ and the electric-displacement field ${}^3\mathbf{d}$ which arise in the piezoelectric plate under the applied loads.

In what follows a decomposition of the three-dimensional Euclidean space is adopted: as an example, the displacement field ${}^3\mathbf{s} = (s_1, s_2, s_3)$ is represented by the couple $(\mathbf{s}, s)$, where $\mathbf{s} = (s_1, s_2)$ and $s = s_3$. An analogous representation is used for the electric field ${}^3\mathbf{e} = (\mathbf{e}, e)$ and the electric-displacement field ${}^3\mathbf{d} = (\mathbf{d}, d)$. As a consequence of the previous decomposition, the symmetric stress tensor ${}^3\mathbf{T}$ is represented by the triplet $(\mathbf{T}, \boldsymbol{\tau}, \sigma)$, where $\sigma = T_{33}$, $\boldsymbol{\tau} = (T_{13}, T_{23})$ and $\mathbf{T}$ is the 2×2 matrix $(T_{11}\ T_{12} \| T_{12}\ T_{22})$. Analogously, the symmetric strain tensor ${}^3\mathbf{E}$ is represented by the triplet $(\mathbf{E}, \gamma/2, \varepsilon)$.

Moreover, a dimensionless thickness variable is introduced:

$$\zeta = \frac{x_3}{h} \tag{2}$$

The Legendre polynomials in the interval $[-1/2 \ldots 1/2]$, up to a multiplicative constant, are given by:

$$
\begin{aligned}
&\mathscr{L}_0(\zeta) = 1 \\
&\mathscr{L}_1(\zeta) = \zeta \\
&\mathscr{L}_2(\zeta) = 6\zeta^2 - 1/2 \\
&\mathscr{L}_3(\zeta) = \zeta(10\zeta^2 - 3/2) \\
&\mathscr{L}_4(\zeta) = 70\zeta^4 - 15\zeta^2 + 3/8
\end{aligned} \tag{3}
$$

It is noted that a normalization different from the standard one for these polynomials was chosen, in order to get simpler expressions for the energy functionals of the plate models. In what follows, 1 and ζ are used instead of $\mathscr{L}_0$ and $\mathscr{L}_1$, respectively. Moreover, the following auxiliary polynomials will be used:

$$
\begin{aligned}
&\mathscr{P}_1(\zeta) = \zeta^2 - 1/4 \\
&\mathscr{P}_2(\zeta) = \zeta(\zeta^2 - 1/4) \\
&\mathscr{P}_3(\zeta) = \zeta(4\zeta^2 - 3) \\
&\mathscr{P}_4(\zeta) = \zeta^4 - (3/10)\zeta^2 + 1/80 \\
&\mathscr{P}_5(\zeta) = \zeta^4 - \zeta^2/2 + 1/16 \\
&\mathscr{P}_6(\zeta) = \zeta(\zeta^4 - \zeta^2/2 + 1/16)
\end{aligned} \tag{4}
$$

It is well known that the loads applied to a piezoelectric plate enter the governing equations through some resultants. In Section 4, concerning piezoelectric plates in bending, the following load resultants will be considered:

$$
\begin{aligned}
\mathbf{m} &= \int_{-h/2}^{h/2} x_3 \mathbf{b}\, \mathrm{d}x_3 + (h/2)(\mathbf{p}^+ - \mathbf{p}^-) \\
\mathbf{n} &= \int_{-h/2}^{h/2} h\mathscr{L}_3(x_3/h)\mathbf{b}\, \mathrm{d}x_3 + (h/2)(\mathbf{p}^+ - \mathbf{p}^-) \\
q &= \int_{-h/2}^{h/2} b\, \mathrm{d}x_3 + p^+ + p^- \\
r &= \int_{-h/2}^{h/2} \mathscr{L}_2(x_3/h) b\, \mathrm{d}x_3 + p^+ + p^- \\
c &= \int_{-h/2}^{h/2} \mu\, \mathrm{d}x_3 + \omega^+ + \omega^- \\
t &= \int_{-h/2}^{h/2} \mathscr{L}_2(x_3/h)\mu\, \mathrm{d}x_3 + \omega^+ + \omega^- \\
g &= \int_{-h/2}^{h/2} \mathscr{L}_4(x_3/h)\mu\, \mathrm{d}x_3 + \omega^+ + \omega^-
\end{aligned} \tag{5}
$$

Analogously, in Section 5, concerning the stretching behaviour of piezoelectric plates, the following load resultants will be considered:

$$
\begin{aligned}
\mathbf{f} &= \int_{-h/2}^{h/2} \mathbf{b}\,\mathrm{d}x_3 + \mathbf{p}^+ + \mathbf{p}^- \\
\mathbf{g} &= \int_{-h/2}^{h/2} (h^2/12)\mathscr{L}_2(x_3/h)\mathbf{b}\,\mathrm{d}x_3 + (h^2/12)(\mathbf{p}^+ + \mathbf{p}^-) \\
a &= \int_{-h/2}^{h/2} x_3 b\,\mathrm{d}x_3 + (h/2)(p^+ - p^-) \\
\theta &= \int_{-h/2}^{h/2} x_3 \mu\,\mathrm{d}x_3 + (h/2)(\omega^+ - \omega^-) \\
\kappa &= \int_{-h/2}^{h/2} h\mathscr{L}_3(x_3/h)\mu\,\mathrm{d}x_3 + (h/2)(\omega^+ - \omega^-)
\end{aligned}
\tag{6}
$$

3. Rational derivation of piezoelectric plate theories

In this section, the variational approach, adopted to derive some higher-order piezoelectric plate models from the three-dimensional constrained piezoelectricity, is briefly described. Different variational formulations of the linear theory of piezoelectricity are available: a comprehensive list is reported by Bisegna and Maceri (1998). Here a formulation based on the potential energy functional is adopted, especially useful in the derivation of finite-element formulations. When it is specialized to the case of a transversely isotropic piezoelectric plate, the potential energy functional becomes:

$$
\begin{aligned}
\mathscr{E} = \frac{1}{2}\int_{\mathscr{P}} &[2c_{66}||\widetilde{\nabla}\mathbf{s}||^2 + c_{12}(\operatorname{div}\mathbf{s})^2 - \varepsilon_{11}||\nabla\phi||^2 + c_{44}||\mathbf{s}' + \nabla s||^2 + c_{33}(s')^2 + 2c_{13}s'\operatorname{div}\mathbf{s} - \varepsilon_{33}(\phi')^2 \\
&+ 2e_{15}\nabla\phi\cdot(\mathbf{s}' + \nabla s) + 2e_{31}\phi'\operatorname{div}\mathbf{s} + 2e_{33}s'\phi']\,\mathrm{d}v - \int_{\mathscr{P}}(\mathbf{b}\cdot\mathbf{s} + bs - \mu\phi)\,\mathrm{d}v \\
&- \int_{\Omega}(\mathbf{p}^+\cdot\mathbf{s}^+ + \mathbf{p}^-\cdot\mathbf{s}^- + p^+s^+ + p^-s^- - \omega^+\phi^+ - \omega^-\phi^-)\,\mathrm{d}a
\end{aligned}
\tag{7}
$$

Here ∇ and div denote, respectively, the gradient and divergence operators taken with respect to the x_1, x_2 variables only, a prime denotes the differentiation with respect to x_3, $\widetilde{\nabla}$ is the symmetric part of the gradient, $||\cdot||$ denotes the norm, and $\mathbf{s}^{\pm}$, $s^{\pm}$, $\phi^{\pm}$ denote, respectively, the values of the fields $\mathbf{s}$, s, ϕ computed at $x_3 = \pm h/2$. According to the assumed boundary conditions, $\mathbf{s}$, s and ϕ vanish on the lateral boundary of the plate. The equations governing the piezoelectric problem for the plate, regarded as a three-dimensional body, can be easily obtained as stationary conditions for the functional $\mathscr{E}$. Of course, the relevant differential problem is defined in the three-dimensional region $\mathscr{P}$: with very few exceptions, it turns out to be untractable from an analytical point of view, and difficult even from a numerical point of view.

On the other hand, piezoelectric plate theories are based on simplifying assumptions, defining how the displacement components and electric potential vary through the plate thickness. These assumptions are introduced in order to reduce the three-dimensional problem to a two-dimensional one. Indeed, in this paper, the representations of the displacement and electric potential fields are deduced as a consequence of hypotheses on the strain and electric fields, respectively. These hypotheses, in turn, are paired with corresponding relaxation conditions on the stress and electric-displacement fields. Of course, each theory is characterized by different assumptions, regarded here as internal constraints imposed on the strain, stress, electric and electric-displacement fields. The theory of Lagrange multipliers is employed to build up a constrained potential-energy functional for each theory. It is well known that the constraints on the stress and electric-displacement fields make reactive strain and electric fields arise; as a consequence, it is necessary to carefully distinguish between the total strain and electric fields (satisfying the compatibility equations (13) and denoted by a subscript t) and the constitutive strain and electric fields (satisfying the constitutive equations (15) and denoted by a subscript c). Analogously, the constraints on the strain and electric fields make reactive stress and electric-displacement fields arise; as a consequence, the stress and electric-displacement fields are split: the total stress and electric-displacement fields (denoted by a subscript t) satisfy the equilibrium equations (9) and (10) whereas the constitutive stress and electric-displacement fields (denoted by a subscript c) satisfy the constitutive equations (15). The relations among such fields, reported in Eqs. (14) and (16), are straightforwardly obtained as a consequence of the machinery adopted.

The following constrained potential-energy functional (Bisegna and Sacco, 1997; Bisegna, 1997) is adopted:

$$\begin{aligned}\mathscr{F} = \frac{1}{2}\int_{\mathscr{P}} \Big[& 2c_{66}\|\widetilde{\nabla}\mathbf{s}\|^2 + c_{12}(\operatorname{div}\mathbf{s})^2 - \varepsilon_{11}\|\nabla\phi\|^2 + c_{44}\|\mathbf{s}' + \nabla s - \mathbf{H}_\tau^*\boldsymbol{\lambda}_\tau\|^2 + c_{33}(s' - H_\sigma^*\lambda_\sigma)^2 \\ & + 2c_{13}(s' - H_\sigma^*\lambda_\sigma)\operatorname{div}\mathbf{s} - \varepsilon_{33}(\phi' + H_d^*\lambda_d)^2 + 2e_{15}\nabla\phi\cdot(\mathbf{s}' + \nabla s - \mathbf{H}_\tau^*\boldsymbol{\lambda}_\tau) + 2e_{31}(\phi' + H_d^*\lambda_d)\operatorname{div}\mathbf{s} \\ & + 2e_{33}(s' - H_\sigma^*\lambda_\sigma)(\phi' + H_d^*\lambda_d)\Big]\,\mathrm{d}v - \int_{\mathscr{P}} [\boldsymbol{\theta}_\gamma\cdot\mathbf{G}_\gamma(\mathbf{s}' + \nabla s) + \theta_\varepsilon G_\varepsilon s' + \theta_e G_e \phi']\,\mathrm{d}v \\ & - \int_{\mathscr{P}} (\mathbf{b}\cdot\mathbf{s} + bs - \mu\phi)\,\mathrm{d}v - \int_{\Omega} (\mathbf{p}^+\cdot\mathbf{s}^+ + \mathbf{p}^-\cdot\mathbf{s}^- + p^+s^+ + p^-s^- - \omega^+\phi^+ - \omega^-\phi^-)\,\mathrm{d}a \end{aligned} \tag{8}$$

Here $\mathbf{H}_\tau$, H_σ, H_d, and $\mathbf{G}_\gamma$, G_ε, G_e are linear differential constraint operators, respectively acting on the constitutive transversal shear stress $\boldsymbol{\tau}_c$, the constitutive transversal normal stress σ_c, the constitutive transversal electric displacement d_c, the total transversal shear strain $\boldsymbol{\gamma}_t$, the total transversal elongation ε_t, the total transversal electric field e_t. The adjoints of these operators are denoted by a superscript $*$. The fields $\boldsymbol{\lambda}_\tau$, λ_σ, λ_d and $\boldsymbol{\theta}_\gamma$, θ_ε, θ_e are Lagrange multipliers. The meaning of the functional $\mathscr{F}$ can be easily understood by its stationary conditions, which are reported in what follows. For the sake of physical evidence, some positions, shown in Eqs. (13)–(16), were made in writing these stationary conditions.

The stationary conditions for $\mathscr{F}$ with respect to $\mathbf{s}$, s and ϕ imply the equilibrium equations:

$$\begin{aligned} & -\operatorname{div}\mathbf{T}_t - \boldsymbol{\tau}_t' = \mathbf{b} \\ & -\operatorname{div}\boldsymbol{\tau}_t - \sigma_t' = b \\ & -\operatorname{div}\mathbf{d}_t - d_t' = -\mu \end{aligned} \tag{9}$$

which hold in $\mathscr{P}$, and the boundary equilibrium equations on the upper and lower surfaces of the plate:

$$\begin{aligned} & \pm\boldsymbol{\tau}_t^\pm = \mathbf{p}^\pm \\ & \pm\sigma_t^\pm = p^\pm \\ & \pm d_t^\pm = -\omega^\pm \end{aligned} \tag{10}$$

The stationary conditions for $\mathscr{F}$ with respect to $\boldsymbol{\theta}_\gamma$, θ_ε and θ_e give, respectively, the constraint equations:

$$\begin{aligned} & \mathbf{G}_\gamma\boldsymbol{\gamma}_t = 0 \\ & G_\varepsilon\varepsilon_t = 0 \\ & G_e e_t = 0 \end{aligned} \tag{11}$$

whereas the stationary conditions for $\mathscr{F}$ with respect to $\boldsymbol{\lambda}_\tau$, λ_σ and λ_d give, respectively, the constraint equations:

$$
\begin{aligned}
&\mathbf{H}_\tau \boldsymbol{\tau}_c = 0\\
&H_\sigma \sigma_c = 0\\
&H_d d_c = 0
\end{aligned}
\tag{12}
$$

Both the sets of constraint equations hold in $\mathscr{P}$.

The positions made in writing the stationary conditions for $\mathscr{F}$ are now reported (Bisegna and Sacco, 1997; Bisegna, 1997). The total strain and electric fields, respectively derived from the displacement and electric-potential fields, are given by:

$$
\begin{aligned}
&\mathbf{E}_t = \widetilde{\nabla}\mathbf{s}\\
&\gamma_t = \mathbf{s}' + \nabla s\\
&\varepsilon_t = s'\\
&\mathbf{e}_t = -\nabla\phi\\
&e_t = -\phi'
\end{aligned}
\tag{13}
$$

The constitutive strain and electric fields, which differ from the total ones due to the reactive fields arising as a consequence of the constraints on the stress and electric-displacement fields, are given by:

$$
\begin{aligned}
&\mathbf{E}_c = \mathbf{E}_t\\
&\gamma_c = \gamma_t - \mathbf{H}_\tau^* \boldsymbol{\lambda}_\tau\\
&\varepsilon_c = \varepsilon_t - H_\sigma^* \lambda_\sigma\\
&\mathbf{e}_c = \mathbf{e}_t\\
&e_c = e_t - H_d^* \lambda_d
\end{aligned}
\tag{14}
$$

The constitutive stress and electric-displacement fields, related to the constitutive strain and electric fields according to the piezoelectric constitutive equations, are given by:

$$
\begin{aligned}
&\mathbf{T}_c = 2c_{66}\mathbf{E}_c + c_{12}\mathbf{I}\,\mathrm{tr}\mathbf{E}_c + c_{13}\varepsilon_c\mathbf{I} - e_{31}e_c\mathbf{I}\\
&\boldsymbol{\tau}_c = c_{44}\gamma_c - e_{15}\mathbf{e}_c\\
&\sigma_c = c_{33}\varepsilon_c + c_{13}\,\mathrm{tr}\mathbf{E}_c - e_{33}e_c\\
&\mathbf{d}_c = e_{15}\gamma_c + \varepsilon_{11}\mathbf{e}_c\\
&d_c = e_{31}\,\mathrm{tr}\mathbf{E}_c + e_{33}\varepsilon_c + \varepsilon_{33}e_c
\end{aligned}
\tag{15}
$$

where $\mathbf{I}$ is the identity tensor and tr denotes the trace operator. Finally, the total stress and electric-displacement fields, which differ from the constitutive ones due to the reactive fields arising as a consequence of the constraints on the strain and electric fields, are given by:

$$
\begin{aligned}
&\mathbf{T}_t = \mathbf{T}_c\\
&\boldsymbol{\tau}_t = \boldsymbol{\tau}_c - \mathbf{G}_\gamma^* \boldsymbol{\theta}_\gamma\\
&\sigma_t = \sigma_c - G_\varepsilon^* \theta_\varepsilon\\
&\mathbf{d}_t = \mathbf{d}_c\\
&d_t = d_c - G_e^* \theta_e
\end{aligned}
\tag{16}
$$

These positions emphasize the physical meaning of the stationary conditions (9)–(12). As it was anticipated at the beginning of this section, it turns out that the equilibrium equations (9) and (10) involve the total stress and electric-displacement fields, the constraint operators $\mathbf{G}_\gamma$, G_ε and G_e act on the total strain and electric fields, and the constraint operators $\mathbf{H}_\tau$, H_σ and H_d act on the constitutive stress and electric-displacement fields.

4. Models for the flexural behaviour of piezoelectric plates

In this section, different models for the flexural behaviour of piezoelectric plates are rationally derived from the three-dimensional piezoelectricity. For each model, both the assumption on the kinematics and the assumption on the electric potential are specified. Needless to say that, when representing the displacement and electric potential fields for a plate in flexure, the $\mathbf{s}$ field has to be an odd function with respect to the thickness variable, whereas the s and ϕ fields have to be even. Three different kinematical assumptions are considered here: the Kirchhoff–Love kinematics, the Reissner–Mindlin one, and the Lo–Christensen–Wu one. Analogously, three different assumptions on the electric potential are investigated: it is assumed to be constant, or quadratic or biquadratic in the thickness variable. For the sake of brevity, not all the 3×3 combinations are adopted in order to build up models for piezoelectric plates. As an example, it is well known that the Kirchhoff–Love kinematics together with a constant electric potential gives rise to no electromechanical coupling (Maugin and Attou, 1990), and hence this combination is left out. The assumptions on the kinematics and electric potential are here regarded as a consequence of constraints on the strain and electric fields, respectively. In turn, these constraints are paired with exactly corresponding relaxation conditions on the stress and electric displacement fields.

In order to obtain the energy functional and the equations governing each plate model, the relevant constraint operators, specified in Table 1, are introduced into the constrained variational formulation (8). Then, the stationary conditions for this functional with respect to the Lagrange multipliers are enforced, and the obtained constraint equations are used in order to represent the displacement and electric potential fields, also reported in Table 1. Finally, they are substituted into the constrained functional. In this way, a pure (i.e., not depending on the Lagrange multipliers) energy functional is obtained, after the integration in the thickness variable x_3 has been performed.

It is interesting to compare the energy functionals and the governing equations of different plate models. It can be observed that the auxiliary material constants introduced in Eq. (1) appear in lower-order models, but do not affect the corresponding terms of higher-order models. This is because such constants arise as a

Table 1
Higher-order plate models for flexure[a]

	Constraint on ε_t, $G_\varepsilon \varepsilon_t = 0$	Constraint on γ_t, $\mathbf{G}_\gamma \gamma_t = 0$	Constraint on e_t, $G_e e_t = 0$	Constraint on σ_c, $H_\sigma \sigma_c = 0$,	Constraint on τ_c, $\mathbf{H}_\tau \tau_c = 0$	Constraint on d_c, $H_d d_c = 0$
KL/Q	$\varepsilon_t = 0$	$\gamma_t = 0$	$e_t'' = 0$	$\sigma_c = 0$	$\tau_c = 0$	$d_c'' = 0$
RM/C	$\varepsilon_t = 0$	$\gamma_t' = 0$	$e_t = 0$	$\sigma_c = 0$	$\tau_c' = 0$	$d_c = 0$
RM/Q	$\varepsilon_t = 0$	$\gamma_t' = 0$	$e_t'' = 0$	$\sigma_c = 0$	$\tau_c' = 0$	$d_c'' = 0$
LCW/Q	$\varepsilon_t'' = 0$	$\gamma_t''' = 0$	$e_t'' = 0$	$\sigma_c'' = 0$	$\tau_c''' = 0$	$d_c'' = 0$
LCW/B	$\varepsilon_t'' = 0$	$\gamma_t''' = 0$	$e_t'''' = 0$	$\sigma_c'' = 0$	$\tau_c''' = 0$	$d_c'''' = 0$

	$\mathbf{s}$	s	ϕ
KL/Q	$-\nabla w(x_1, x_2) h\zeta$	$w(x_1, x_2)$	$\eta(x_1, x_2) + \upsilon(x_1, x_2)\mathscr{L}_2$
RM/C	$\boldsymbol{\varphi}(x_1, x_2) h\zeta$	$w(x_1, x_2)$	$\eta(x_1, x_2)$
RM/Q	$\boldsymbol{\varphi}(x_1, x_2) h\zeta$	$w(x_1, x_2)$	$\eta(x_1, x_2) + \upsilon(x_1, x_2)\mathscr{L}_2$
LCW/Q	$\boldsymbol{\varphi}(x_1, x_2) h\zeta + \boldsymbol{\psi}(x_1, x_2) h\mathscr{L}_3$	$w(x_1, x_2) + \alpha(x_1, x_2)\mathscr{L}_2$	$\eta(x_1, x_2) + \upsilon(x_1, x_2)\mathscr{L}_2$
LCW/B	$\boldsymbol{\varphi}(x_1, x_2) h\zeta + \boldsymbol{\psi}(x_1, x_2) h\mathscr{L}_3$	$w(x_1, x_2) + \alpha(x_1, x_2)\mathscr{L}_2$	$\eta(x_1, x_2) + \upsilon(x_1, x_2)\mathscr{L}_2 + \beta(x_1, x_2)\mathscr{L}_4$

[a] Assumptions on the strain and electric fields. Corresponding relaxation conditions on the stress and electric displacement fields. Representations of the displacement and electric potential fields.

consequence of the relaxation conditions on the stress and electric displacement terms, which are "relaxed" in higher-order models, according to the weaker assumptions on the strain and electric fields adopted in the latter models.

4.1. Kirchhoff–Love-type model with quadratic potential – KL/Q

The model dealt with in this section is based on the Kirchhoff–Love kinematics and assumes a quadratic variation in the thickness for the electric potential. It was recently presented by Yang (1999) and adopted in the modelization of elastic plates with partially electroded actuators. In this model the transversal elongation and shear strain vanish, whereas the transversal electric field is linear in the thickness, as shown in Table 1. Accordingly, both the transversal normal and shear stress are assumed to vanish. On the other hand, the transversal electric displacement is assumed to vary linearly in the thickness. After these constraints are enforced, the following energy functional is obtained, depending on one mechanical unknown, the deflection w, and two electric unknown (η and υ):

$$\begin{aligned}\mathscr{F} = {} & \frac{h^3}{24}\bar{c}_{11}\int_\Omega [(1-\bar{v})||\nabla\nabla w||^2 + \bar{v}(\Delta w)^2]\,\mathrm{d}a - \frac{h}{2}\bar{\varepsilon}_{11}\int_\Omega \left[||\nabla\eta||^2 + \frac{1}{5}||\nabla\upsilon||^2\right]\mathrm{d}a - \frac{6}{h}\bar{\varepsilon}_{33}\int_\Omega \upsilon^2\,\mathrm{d}a \\ & - h\bar{e}_{31}\int_\Omega \upsilon\Delta w\,\mathrm{d}a - \int_\Omega(-\mathbf{m}\cdot\nabla w + qw - c\eta - t\upsilon)\,\mathrm{d}a \end{aligned} \tag{17}$$

The stationary conditions for this functional with respect to w, η and υ supply the following mechanical and electric equilibrium equations:

$$\frac{h^3}{12}\bar{c}_{11}\Delta\Delta w - h\bar{e}_{31}\Delta\upsilon = \operatorname{div}\mathbf{m} + q \quad h\bar{\varepsilon}_{11}\Delta\eta = -c \quad \frac{h}{5}\bar{\varepsilon}_{11}\Delta\upsilon - h\bar{e}_{31}\Delta w - \frac{12}{h}\bar{\varepsilon}_{33}\upsilon = -t \tag{18}$$

It appears that the low-order electric unknown η is uncoupled from the remaining unknowns.

This model is governed by the bilaplacian operators, and the energy functional involves second derivatives of the unknown function w. This issue has an important consequence in the development of finite-element formulations, since it implies that C^1 shape functions should be adopted.

4.2. Reissner–Mindlin-type model with constant potential – RM/C

This model is based on the Reissner–Mindlin kinematics and assumes a constant electric potential in the thickness: hence, it involves only linear expansions in the thickness variable. Such assumptions were firstly introduced by Mindlin (1952), then considered by Dökmeci (1980). They imply that the transversal elongation and electric field vanish, whereas the transversal shear strain is constant in the thickness, as shown in Table 1. When coupled with the corresponding relaxation conditions (Bisegna, 1997) the following energy functional is obtained, depending on three mechanical unknowns, the deflection w and the rotations of the fibres orthogonal to the middle plane $\boldsymbol{\varphi} = (\varphi_1, \varphi_2)$, and one electric unknown, the potential on the plate middle plane η:

$$\begin{aligned}\mathscr{F} = {} & \frac{h^3}{24}\hat{c}_{11}\int_\Omega [(1-\hat{v})||\widetilde{\nabla}\boldsymbol{\varphi}||^2 + \hat{v}(\operatorname{div}\boldsymbol{\varphi})^2]\,\mathrm{d}a + \frac{h}{2}c_{44}\int_\Omega ||\boldsymbol{\varphi} + \nabla w||^2\,\mathrm{d}a \\ & + he_{15}\int_\Omega \nabla\eta\cdot(\boldsymbol{\varphi} + \nabla w)\,\mathrm{d}a - \frac{h}{2}\varepsilon_{11}\int_\Omega ||\nabla\eta||^2\,\mathrm{d}a - \int_\Omega(\mathbf{m}\cdot\boldsymbol{\varphi} + qw - c\eta)\,\mathrm{d}a \end{aligned} \tag{19}$$

The stationary conditions for this functional with respect to $\boldsymbol{\varphi}$, w and η supply the following mechanical and electric equilibrium equations:

$$
\begin{aligned}
&-\frac{h^3}{12}\hat{c}_{11}\operatorname{div}[(1-\hat{\nu})\widetilde{\nabla}\boldsymbol{\varphi}+\hat{\nu}\mathbf{I}\operatorname{div}\boldsymbol{\varphi}]+hc_{44}(\boldsymbol{\varphi}+\nabla w)+he_{15}\nabla\eta=\mathbf{m}\\
&-hc_{44}\operatorname{div}(\boldsymbol{\varphi}+\nabla w)-he_{15}\Delta\eta=q\\
&-he_{15}\operatorname{div}(\boldsymbol{\varphi}+\nabla w)+h\varepsilon_{11}\Delta\eta=-c
\end{aligned}
\tag{20}
$$

The appearance of the constant $\hat{c}_{11}$, due to both the relaxation conditions $\sigma_c = 0$ and $d_c = 0$ is pointed out: the correct stiffness of a piezoelectric plate in bending is proportional to this constant (Bisegna and Maceri, 1996b).

A locking-free finite-element formulation based on this model was recently developed by Auricchio et al. (2000).

4.3. Reissner–Mindlin-type model with quadratic potential – RM/Q

This model summarizes the merits of the previous two models KL/Q and RM/C, since it is based on the Reissner–Mindlin kinematics and assumes a quadratic variation in the thickness for the electric potential (Caruso, 2000). The relevant constraint operators are shown in Table 1. The governing energy functional, depending on three mechanical fields (w and $\boldsymbol{\varphi} = (\varphi_1, \varphi_2)$) and two fields of electric nature (η and υ), turns out to be:

$$
\begin{aligned}
\mathscr{F} = {}&\frac{h^3}{24}\bar{c}_{11}\int_\Omega[(1-\bar{\nu})||\widetilde{\nabla}\boldsymbol{\varphi}||^2+\bar{\nu}(\operatorname{div}\boldsymbol{\varphi})^2]\,\mathrm{d}a+\frac{h}{2}c_{44}\int_\Omega||\boldsymbol{\varphi}+\nabla w||^2\,\mathrm{d}a+he_{15}\int_\Omega\nabla\eta\cdot(\boldsymbol{\varphi}+\nabla w)\,\mathrm{d}a\\
&-\frac{h}{2}\varepsilon_{11}\int_\Omega||\nabla\eta||^2\,\mathrm{d}a-\frac{h}{10}\bar{\varepsilon}_{11}\int_\Omega||\nabla\upsilon||^2\,\mathrm{d}a-\frac{6}{h}\bar{\varepsilon}_{33}\int_\Omega\upsilon^2\,\mathrm{d}a+h\bar{e}_{31}\int_\Omega\upsilon\operatorname{div}\boldsymbol{\varphi}\,\mathrm{d}a\\
&-\int_\Omega(\mathbf{m}\cdot\boldsymbol{\varphi}+qw-c\eta-t\upsilon)\,\mathrm{d}a
\end{aligned}
\tag{21}
$$

Its stationary conditions with respect to $\boldsymbol{\varphi}$, w, η and υ supply the following mechanical and electric equilibrium equations:

$$
\begin{aligned}
&-\frac{h^3}{12}\bar{c}_{11}\operatorname{div}[(1-\bar{\nu})\widetilde{\nabla}\boldsymbol{\varphi}+\bar{\nu}\mathbf{I}\operatorname{div}\boldsymbol{\varphi}]-h\bar{e}_{31}\nabla\upsilon+hc_{44}(\boldsymbol{\varphi}+\nabla w)+he_{15}\nabla\eta=\mathbf{m}\\
&-hc_{44}\operatorname{div}(\boldsymbol{\varphi}+\nabla w)-he_{15}\Delta\eta=q\\
&-he_{15}\operatorname{div}(\boldsymbol{\varphi}+\nabla w)+h\varepsilon_{11}\Delta\eta=-c\\
&\frac{h}{5}\bar{\varepsilon}_{11}\Delta\upsilon+h\bar{e}_{31}\operatorname{div}\boldsymbol{\varphi}-\frac{12}{h}\bar{\varepsilon}_{33}\upsilon=-t
\end{aligned}
\tag{22}
$$

4.4. Lo–Christensen–Wu-type model with quadratic potential – LCW/Q

With respect to the model RM/Q, this model improves the kinematics, since it is based on the Lo–Christensen–Wu representation instead of the Reissner–Mindlin one. The relaxation conditions are accordingly modified: as an example, the transversal normal stress is now not assumed to vanish, but it is assumed to be linear in the thickness. The constraint operators are shown in Table 1. The governing energy functional depends on six mechanical fields ($\boldsymbol{\varphi} = (\varphi_1, \varphi_2)$, $\boldsymbol{\psi} = (\psi_1, \psi_2)$, w and α) and two fields of electric nature (η and υ); it turns out to be:

$$\begin{aligned}\mathscr{F} &= \frac{h^3}{24}c_{11}\int_\Omega \left[(1-v)||\widetilde{\nabla}\boldsymbol{\varphi}||^2 + v(\operatorname{div}\boldsymbol{\varphi})^2\right]\mathrm{d}a + \frac{h^3}{56}\hat{c}_{11}\int_\Omega \left[(1-\hat{v})||\widetilde{\nabla}\boldsymbol{\psi}||^2 + \hat{v}(\operatorname{div}\boldsymbol{\psi})^2\right]\mathrm{d}a \\ &\quad + \frac{h}{2}c_{44}\int_\Omega \left[||\boldsymbol{\varphi}+\boldsymbol{\psi}+\nabla w||^2 + 5||\boldsymbol{\psi}||^2 + \frac{1}{5}||\nabla\alpha||^2 + 2\boldsymbol{\psi}\cdot\nabla\alpha\right]\mathrm{d}a + he_{15}\int_\Omega \left[\nabla\eta\cdot(\boldsymbol{\varphi}+\boldsymbol{\psi}+\nabla w)\right. \\ &\qquad \left. + \nabla\upsilon\cdot\left(\boldsymbol{\psi}+\frac{1}{5}\nabla\alpha\right)\right]\mathrm{d}a - \frac{h}{2}\varepsilon_{11}\int_\Omega \left[||\nabla\eta||^2 + \frac{1}{5}||\nabla\upsilon||^2\right]\mathrm{d}a + \frac{6}{h}c_{33}\int_\Omega \alpha^2\,\mathrm{d}a - \frac{6}{h}\varepsilon_{33}\int_\Omega \upsilon^2\,\mathrm{d}a + hc_{13} \\ &\quad \times \int_\Omega \alpha\operatorname{div}\boldsymbol{\varphi}\,\mathrm{d}a + he_{31}\int_\Omega \upsilon\operatorname{div}\boldsymbol{\varphi}\,\mathrm{d}a + \frac{12}{h}e_{33}\int_\Omega \alpha\upsilon\,\mathrm{d}a - \int_\Omega (\mathbf{m}\cdot\boldsymbol{\varphi} + \mathbf{n}\cdot\boldsymbol{\psi} + qw + r\alpha - c\eta - t\upsilon)\,\mathrm{d}a\end{aligned} \tag{23}$$

Accordingly, the mechanical and electric equilibrium equations turn out to be:

$$\begin{aligned}&-\frac{h^3}{12}c_{11}\operatorname{div}[(1-v)\widetilde{\nabla}\boldsymbol{\varphi} + v\mathbf{I}\operatorname{div}\boldsymbol{\varphi}] - hc_{13}\nabla\alpha - he_{31}\nabla\upsilon + hc_{44}(\boldsymbol{\varphi}+\boldsymbol{\psi}+\nabla w) + he_{15}\nabla\eta = \mathbf{m} \\ &-\frac{h^3}{28}\hat{c}_{11}\operatorname{div}[(1-\hat{v})\widetilde{\nabla}\boldsymbol{\psi} + \hat{v}\mathbf{I}\operatorname{div}\boldsymbol{\psi}] + hc_{44}(\boldsymbol{\varphi}+6\boldsymbol{\psi}+\nabla w+\nabla\alpha) + he_{15}(\nabla\eta+\nabla\upsilon) = \mathbf{n} \\ &-hc_{44}\operatorname{div}(\boldsymbol{\varphi}+\boldsymbol{\psi}+\nabla w) - he_{15}\Delta\eta = q \\ &-hc_{44}\operatorname{div}\left(\boldsymbol{\psi}+\frac{1}{5}\nabla\alpha\right) - \frac{h}{5}e_{15}\Delta\upsilon + hc_{13}\operatorname{div}\boldsymbol{\varphi} + \frac{12}{h}c_{13}\alpha + \frac{12}{h}e_{33}\upsilon = r \\ &-he_{15}\operatorname{div}(\boldsymbol{\varphi}+\boldsymbol{\psi}+\nabla w) + h\varepsilon_{11}\Delta\eta = -c \\ &-he_{15}\operatorname{div}\left(\boldsymbol{\psi}+\frac{1}{5}\nabla\alpha\right) + \frac{h}{5}\varepsilon_{11}\Delta\upsilon + he_{31}\operatorname{div}\boldsymbol{\varphi} + \frac{12}{h}e_{33}\alpha - \frac{12}{h}\varepsilon_{33}\upsilon = -t\end{aligned} \tag{24}$$

4.5. Lo–Christensen–Wu-type model with biquadratic potential – LCW/B

This model improves the model LCW/Q as to the electric potential, now assumed to have a biquadratic law of variation in the thickness. The relaxation conditions are accordingly modified. The constraint operators are shown in Table 1. The governing energy functional depends on six mechanical fields ($\boldsymbol{\varphi} = (\varphi_1, \varphi_2)$, $\boldsymbol{\psi} = (\psi_1, \psi_2)$, w and α) and three fields of electric nature (η, υ, and β); it turns out to be:

$$\begin{aligned}\mathscr{F} &= \frac{h^3}{24}c_{11}\int_\Omega [(1-v)||\widetilde{\nabla}\boldsymbol{\varphi}||^2 + v(\operatorname{div}\boldsymbol{\varphi})^2]\,\mathrm{d}a + \frac{h^3}{56}\overline{c}_{11}\int_\Omega [(1-\overline{v})||\widetilde{\nabla}\boldsymbol{\psi}||^2 + \overline{v}(\operatorname{div}\boldsymbol{\psi})^2]\,\mathrm{d}a \\ &\quad + \frac{h}{2}c_{44}\int_\Omega \left[||\boldsymbol{\varphi}+\boldsymbol{\psi}+\nabla w||^2 + 5||\boldsymbol{\psi}||^2 + \frac{1}{5}||\nabla\alpha||^2 + 2\boldsymbol{\psi}\cdot\nabla\alpha\right]\mathrm{d}a + he_{15}\int_\Omega \left[\nabla\eta\cdot(\boldsymbol{\varphi}+\boldsymbol{\psi}+\nabla w)\right. \\ &\qquad \left. + \nabla\upsilon\cdot\left(\boldsymbol{\psi}+\frac{1}{5}\nabla\alpha\right)\right]\mathrm{d}a - \frac{h}{2}\varepsilon_{11}\int_\Omega \left[||\nabla\eta||^2 + \frac{1}{5}||\nabla\upsilon||^2\right]\mathrm{d}a - \frac{h}{18}\overline{\varepsilon}_{11}\int_\Omega ||\nabla\beta||^2\,\mathrm{d}a + \frac{6}{h}c_{33}\int_\Omega \alpha^2\,\mathrm{d}a \\ &\quad - \frac{6}{h}\varepsilon_{33}\int_\Omega (\upsilon+\beta)^2\,\mathrm{d}a + hc_{13}\int_\Omega \alpha\operatorname{div}\boldsymbol{\varphi}\,\mathrm{d}a - \frac{14}{h}\overline{\varepsilon}_{33}\int_\Omega \beta^2\,\mathrm{d}a + he_{31}\int_\Omega (\upsilon+\beta)\operatorname{div}\boldsymbol{\varphi}\,\mathrm{d}a \\ &\quad + h\overline{e}_{31}\int_\Omega \beta\operatorname{div}\boldsymbol{\psi}\,\mathrm{d}a + \frac{12}{h}e_{33}\int_\Omega \alpha(\upsilon+\beta)\,\mathrm{d}a - \int_\Omega (\mathbf{m}\cdot\boldsymbol{\varphi} + \mathbf{n}\cdot\boldsymbol{\psi} + qw + r\alpha - c\eta - t\upsilon - g\beta)\,\mathrm{d}a\end{aligned} \tag{25}$$

As a consequence, four mechanical equilibrium equations and three electric equilibrium equations are obtained as stationary conditions for this functional:

$$
\begin{aligned}
&-\frac{h^3}{12}c_{11}\,\mathrm{div}[(1-v)\widetilde{\nabla}\boldsymbol{\varphi}+v\mathbf{I}\,\mathrm{div}\,\boldsymbol{\varphi}]-hc_{13}\nabla\alpha-he_{31}\nabla(\upsilon+\beta)+hc_{44}(\boldsymbol{\varphi}+\boldsymbol{\psi}+\nabla w)+he_{15}\nabla\eta=\mathbf{m}\\
&-\frac{h^3}{28}\overline{c}_{11}\,\mathrm{div}[(1-\overline{v})\widetilde{\nabla}\boldsymbol{\psi}+\overline{v}\mathbf{I}\,\mathrm{div}\,\boldsymbol{\psi}]-h\overline{e}_{31}\nabla\beta+hc_{44}(\boldsymbol{\varphi}+6\boldsymbol{\psi}+\nabla w+\nabla\alpha)+he_{15}(\nabla\eta+\nabla\upsilon)=\mathbf{n}\\
&-hc_{44}\,\mathrm{div}(\boldsymbol{\varphi}+\boldsymbol{\psi}+\nabla w)-he_{15}\Delta\eta=q\\
&-hc_{44}\,\mathrm{div}\left(\boldsymbol{\psi}+\frac{1}{5}\nabla\alpha\right)-\frac{h}{5}e_{15}\Delta\upsilon+hc_{13}\,\mathrm{div}\,\boldsymbol{\varphi}+\frac{12}{h}c_{13}\alpha+\frac{12}{h}e_{33}(\upsilon+\beta)=r\\
&-he_{15}\,\mathrm{div}(\boldsymbol{\varphi}+\boldsymbol{\psi}+\nabla w)+h\varepsilon_{11}\Delta\eta=-c\\
&-he_{15}\,\mathrm{div}\left(\boldsymbol{\psi}+\frac{1}{5}\nabla\alpha\right)+\frac{h}{5}\varepsilon_{11}\Delta\upsilon+he_{31}\,\mathrm{div}\,\boldsymbol{\varphi}+\frac{12}{h}e_{33}\alpha-\frac{12}{h}\varepsilon_{33}(\upsilon+\beta)=-t\\
&\frac{h}{9}\overline{\varepsilon}_{11}\Delta\beta+he_{31}\,\mathrm{div}\,\boldsymbol{\varphi}+h\overline{e}_{31}\,\mathrm{div}\,\boldsymbol{\psi}+\frac{12}{h}e_{33}\alpha-\frac{12}{h}\varepsilon_{33}\upsilon-\frac{4}{h}(7\overline{\varepsilon}_{33}+3\varepsilon_{33})\beta=-g
\end{aligned}
\tag{26}
$$

At the authors' knowledge, this model was not explicitly considered in the literature.

5. Models for the membranal behaviour of piezoelectric plates

This section deals with a rational derivation from the three-dimensional piezoelectricity of different models for the membranal behaviour of piezoelectric plates.

For each model, both the assumption on the kinematics and the assumption on the electric potential are specified. Differently from the case of flexural behaviour treated in Section 4, now the field $\mathbf{s}$ has to be an even function with respect to the thickness variable, whereas the fields s and ϕ have to be odd. Three different kinematical assumptions are considered here: the classical kinematics, the changing-thickness kinematics without transversal shear stress, and the changing thickness kinematics accounting also for transversal shear stress. In addition, two different assumptions on the electric potential are investigated: it is assumed to be linear or cubic in the thickness variable. For the sake of brevity, only some combinations of such assumptions are investigated. The assumptions on the kinematics and electric potential are regarded as a consequence of constraints on the strain and electric fields, respectively. In turn, these constraints are paired with exactly corresponding relaxation conditions on the stress and electric displacement fields.

The energy functional and the equations governing each plate model are obtained as it was explained in Section 4. The constraint operators and the representations of the displacement and electric potential fields are reported in Table 2.

5.1. Classical model with linear potential – C/L

The model dealt with in this section (Tzou, 1993; Tiersten, 1994; Yang, 1997) is based on the classical kinematics of plates in stretching (i.e., constant in-plane displacement and vanishing out-of-plane displacement); moreover, a linear variation in the thickness for the electric potential is enforced. In other words, this model assumes that the transversal elongation and shear strain vanish, whereas the transversal electric field is constant in the thickness, as shown in Table 2. Accordingly, both the transversal normal and shear stress are assumed to vanish. On the other hand, the transversal electric displacement is assumed to be

Table 2
Higher-order plate models for extension[a]

	Constraint on ε_t, $G_\varepsilon \varepsilon_t = 0$	Constraint on γ_t, $\mathbf{G}_\gamma \gamma_t = 0$	Constraint on e_t, $G_e e_t = 0$	Constraint on σ_c, $H_\sigma \sigma_c = 0$	Constraint on τ_c, $\mathbf{H}_\tau \tau_c = 0$	constraint on d_c, $H_d d_c = 0$
C/L	$\varepsilon_t = 0$	$\gamma_t = 0$	$e'_t = 0$	$\sigma_c = 0$	$\tau_c = 0$	$d'_c = 0$
CT/L	$\varepsilon'_t = 0$	$\gamma_t = 0$	$e'_t = 0$	$\sigma'_c = 0$	$\tau_c = 0$	$d'_c = 0$
CTS/L	$\varepsilon'_t = 0$	$\gamma''_t = 0$	$e'_t = 0$	$\sigma'_c = 0$	$\tau''_c = 0$	$d'_c = 0$
CTS/C	$\varepsilon'_t = 0$	$\gamma''_t = 0$	$e'''_t = 0$	$\sigma'_c = 0$	$\tau''_c = 0$	$d'''_c = 0$
	$\mathbf{s}$	s	ϕ			
C/L	$\mathbf{u}(x_1, x_2)$	0	$\chi(x_1, x_2) h\zeta$			
CT/L	$\mathbf{u}(x_1, x_2) - \frac{h^2}{12}\mathscr{L}_2 \nabla\rho(x_1, x_2)$	$\rho(x_1, x_2) h\zeta$	$\chi(x_1, x_2) h\zeta$			
CTS/L	$\mathbf{u}(x_1, x_2) + \frac{h^2}{12}\mathscr{L}_2 \boldsymbol{\lambda}(x_1, x_2)$	$\rho(x_1, x_2) h\zeta$	$\chi(x_1, x_2) h\zeta$			
CTS/C	$\mathbf{u}(x_1, x_2) + \frac{h^2}{12}\mathscr{L}_2 \boldsymbol{\lambda}(x_1, x_2)$	$\rho(x_1, x_2) h\zeta$	$\chi(x_1, x_2) h\zeta + \xi(x_1, x_2) h\mathscr{L}_3$			

[a] Assumptions on the strain and electric fields. Corresponding relaxation conditions on the stress and electric displacement fields. Representations of the displacement and electric potential fields.

constant in the thickness. After these constraints are enforced, the following energy functional is obtained, depending on two mechanical unknowns, the in-plane displacement $\mathbf{u} = (u_1, u_2)$, and one electric unknown, the transversal electric field $-\chi$:

$$\mathscr{F} = \frac{h}{2}\bar{c}_{11} \int_\Omega [(1-\bar{v})\|\widetilde{\nabla}\mathbf{u}\|^2 + \bar{v}(\operatorname{div}\mathbf{u})^2]\,\mathrm{d}a - \frac{h^3}{24}\bar{\varepsilon}_{11} \int_\Omega \|\nabla\chi\|^2\,\mathrm{d}a - \frac{h}{2}\bar{\varepsilon}_{33} \int_\Omega \chi^2\,\mathrm{d}a$$
$$+ h\bar{e}_{31} \int_\Omega \chi \operatorname{div}\mathbf{u}\,\mathrm{d}a - \int_\Omega (\mathbf{f}\cdot\mathbf{u} - \theta\chi)\,\mathrm{d}a \tag{27}$$

The stationary conditions for this functional with respect to $\mathbf{u}$ and χ supply the following mechanical and electric equilibrium equations:

$$\begin{aligned} &-h\bar{c}_{11}\operatorname{div}[(1-\bar{v})\widetilde{\nabla}\mathbf{u} + \bar{v}\mathbf{I}\operatorname{div}\mathbf{u}] - h\bar{e}_{31}\nabla\chi = \mathbf{f} \\ &\frac{h^3}{12}\bar{\varepsilon}_{11}\Delta\chi + h\bar{e}_{31}\operatorname{div}\mathbf{u} - h\bar{\varepsilon}_{33}\chi = -\theta \end{aligned} \tag{28}$$

A finite-element formulation based on this model was developed by Bisegna and Caruso (2000).

5.2. Changing-thickness model with linear potential – CT/L

This model differs from the model C/L as it takes into account the transversal elongation. Moreover, it assumes a vanishing transversal shear strain. This latter feature distinguishes the present model by a model by Nicotra and Podio-Guidugli (1998), assuming that the transversal shear strain is linked to the transversal elongation. The relaxation conditions enforced in the present model, as usual, exactly correspond to the assumptions on the strain and electric field (Table 2). After these constraints are enforced, the following energy functional is obtained, depending on three mechanical unknowns, the in-plane displacement $\mathbf{u} = (u_1, u_2)$ and the transversal elongation ρ, and one electric unknown, the transversal electric field $-\chi$:

$$
\begin{aligned}
\mathscr{F} = {} & \frac{h}{2} c_{11} \int_\Omega \left[(1-\nu) ||\widetilde{\nabla} \mathbf{u}||^2 + \nu (\operatorname{div} \mathbf{u})^2 \right] \mathrm{d}a + \frac{h^5}{1440} \hat{c}_{11} \int_\Omega \left[(1-\hat{\nu}) ||\nabla\nabla\rho||^2 + \hat{\nu} (\Delta\rho)^2 \right] \mathrm{d}a \\
& - \frac{h^3}{24} \bar{\varepsilon}_{11} \int_\Omega ||\nabla\chi||^2 \, \mathrm{d}a + \frac{h}{2} c_{33} \int_\Omega \rho^2 \, \mathrm{d}a + h c_{13} \int_\Omega \rho \operatorname{div} \mathbf{u} \, \mathrm{d}a - \frac{h}{2} \varepsilon_{33} \int_\Omega \chi^2 \, \mathrm{d}a \\
& + h e_{31} \int_\Omega \chi \operatorname{div} \mathbf{u} \, \mathrm{d}a + h e_{33} \int_\Omega \chi \rho \, \mathrm{d}a - \int_\Omega (\mathbf{f} \cdot \mathbf{u} - \mathbf{g} \cdot \nabla\rho + a\rho - \theta\chi) \, \mathrm{d}a
\end{aligned}
\tag{29}
$$

The stationary conditions for this functional with respect to $\mathbf{u}$, ρ and χ supply the following mechanical and electric equilibrium equations:

$$
\begin{aligned}
& - h c_{11} \operatorname{div}[(1-\nu) \widetilde{\nabla} \mathbf{u} + \nu \mathbf{I} \operatorname{div} \mathbf{u}] - h c_{13} \nabla\rho - h e_{31} \nabla\chi = \mathbf{f} \\
& \frac{h^5}{720} \hat{c}_{11} \Delta\Delta\rho + h c_{13} \operatorname{div} \mathbf{u} + h c_{33} \rho + h e_{33} \chi = a + \operatorname{div} \mathbf{g} \\
& \frac{h^3}{12} \bar{\varepsilon}_{11} \Delta\chi + h e_{31} \operatorname{div} \mathbf{u} + h e_{33} \rho - h \varepsilon_{33} \chi = -\theta
\end{aligned}
\tag{30}
$$

The appearance of the auxiliary material constant c_{11} in the first equation, instead of $\bar{c}_{11}$ appearing in the model C/L, is a consequence of the relaxation condition $\sigma'_c = 0$, which is weaker than the one introduced in the model C/L.

5.3. Changing-thickness model with transversal shear and linear potential – CTS/L

This model improves the model CT/L since it allows for a non-vanishing transversal shear strain. The kinematical assumptions and the related relaxation conditions are reported in Table 2. The energy functional depends on five mechanical unknowns ($\mathbf{u} = (u_1, u_2)$, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ and ρ), and one electric unknown (χ). It turns out to be:

$$
\begin{aligned}
\mathscr{F} = {} & \frac{h}{2} c_{11} \int_\Omega [(1-\nu) ||\widetilde{\nabla} \mathbf{u}||^2 + \nu (\operatorname{div} \mathbf{u})^2] \, \mathrm{d}a + \frac{h^5}{1440} \hat{c}_{11} \int_\Omega [(1-\hat{\nu}) ||\widetilde{\nabla} \boldsymbol{\lambda}||^2 + \hat{\nu} (\operatorname{div} \boldsymbol{\lambda})^2] \, \mathrm{d}a \\
& + \frac{h^3}{24} c_{44} \int_\Omega ||\boldsymbol{\lambda} + \nabla\rho||^2 \, \mathrm{d}a - \frac{h^3}{24} \varepsilon_{11} \int_\Omega ||\nabla\chi||^2 \, \mathrm{d}a + \frac{h^3}{12} e_{15} \int_\Omega (\boldsymbol{\lambda} + \nabla\rho) \cdot \nabla\chi \, \mathrm{d}a + \frac{h}{2} c_{33} \int_\Omega \rho^2 \, \mathrm{d}a \\
& + h c_{13} \int_\Omega \rho \operatorname{div} \mathbf{u} \, \mathrm{d}a - \frac{h}{2} \varepsilon_{33} \int_\Omega \chi^2 \, \mathrm{d}a + h e_{31} \int_\Omega \chi \operatorname{div} \mathbf{u} \, \mathrm{d}a + h e_{33} \int_\Omega \chi\rho \, \mathrm{d}a \\
& - \int_\Omega (\mathbf{f} \cdot \mathbf{u} + \mathbf{g} \cdot \boldsymbol{\lambda} + a\rho - \theta\chi) \, \mathrm{d}a
\end{aligned}
\tag{31}
$$

The governing equations are:

$$
\begin{aligned}
& - h c_{11} \operatorname{div}[(1-\nu) \widetilde{\nabla} \mathbf{u} + \nu \mathbf{I} \operatorname{div} \mathbf{u}] - h c_{13} \nabla\rho - h e_{31} \nabla\chi = \mathbf{f} \\
& - \frac{h^5}{720} \hat{c}_{11} \operatorname{div}[(1-\hat{\nu}) \widetilde{\nabla} \boldsymbol{\lambda} + \hat{\nu} \mathbf{I} \operatorname{div} \boldsymbol{\lambda}] + \frac{h^3}{12} c_{44} (\boldsymbol{\lambda} + \nabla\rho) + \frac{h^3}{12} e_{15} \nabla\chi = \mathbf{g} \\
& - \frac{h^3}{12} c_{44} \operatorname{div}(\boldsymbol{\lambda} + \nabla\rho) - \frac{h^3}{12} e_{15} \Delta\chi + h c_{13} \operatorname{div} \mathbf{u} + h c_{33} \rho + h e_{33} \chi = a \\
& - \frac{h^3}{12} e_{15} \operatorname{div}(\boldsymbol{\lambda} + \nabla\rho) + \frac{h^3}{12} \varepsilon_{11} \Delta\chi + h e_{31} \operatorname{div} \mathbf{u} + h e_{33} \rho - h \varepsilon_{33} \chi = -\theta
\end{aligned}
\tag{32}
$$

5.4. Changing-thickness model with transversal shear and cubic potential – CTS/C

Finally, this model improves the model CTS/L since it allows for an electric potential varying cubically in the thickness variable. The underlying assumptions and the related relaxation conditions are reported in Table 2. The energy functional depends on five mechanical unknowns ($\mathbf{u} = (u_1, u_2)$, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ and ρ), and two electric unknowns (χ and ξ). It turns out to be:

$$\begin{aligned}\mathscr{F} = {} & \frac{h}{2}c_{11}\int_\Omega\left[(1-v)||\widetilde{\nabla}\mathbf{u}||^2 + v(\operatorname{div}\mathbf{u})^2\right]\mathrm{d}a + \frac{h^5}{1440}\bar{c}_{11}\int_\Omega\left[(1-\bar{v})||\widetilde{\nabla}\boldsymbol{\lambda}||^2 + \bar{v}(\operatorname{div}\boldsymbol{\lambda})^2\right]\mathrm{d}a \\ & + \frac{h^3}{24}c_{44}\int_\Omega||\boldsymbol{\lambda} + \nabla\rho||^2\,\mathrm{d}a - \frac{h^3}{24}\varepsilon_{11}\int_\Omega||\nabla\chi||^2\,\mathrm{d}a + \frac{h^3}{12}e_{15}\int_\Omega(\boldsymbol{\lambda} + \nabla\rho)\cdot\nabla\chi\,\mathrm{d}a \\ & - \frac{h^3}{56}\bar{\varepsilon}_{11}\int_\Omega||\nabla\xi||^2\,\mathrm{d}a + \frac{h}{2}c_{33}\int_\Omega\rho^2\,\mathrm{d}a + hc_{13}\int_\Omega\rho\operatorname{div}\mathbf{u}\,\mathrm{d}a - \frac{h}{2}\varepsilon_{33}\int_\Omega(\chi+\xi)^2\,\mathrm{d}a - \frac{5h}{2}\bar{\varepsilon}_{33}\int_\Omega\xi^2\,\mathrm{d}a \\ & + he_{31}\int_\Omega(\chi+\xi)\operatorname{div}\mathbf{u}\,\mathrm{d}a + he_{33}\int_\Omega(\chi+\xi)\rho\,\mathrm{d}a + \frac{h^3}{12}\bar{e}_{31}\int_\Omega\xi\operatorname{div}\boldsymbol{\lambda}\,\mathrm{d}a - \int_\Omega(\mathbf{f}\cdot\mathbf{u} + \mathbf{g}\cdot\boldsymbol{\lambda} + a\rho - \theta\chi - \kappa\xi)\,\mathrm{d}a\end{aligned} \tag{33}$$

As usual, the governing mechanical and electric equilibrium equations are obtained as stationary conditions for this functional with respect to the cited unknowns They result to be:

$$\begin{aligned} & -hc_{11}\operatorname{div}[(1-v)\widetilde{\nabla}\mathbf{u} + v\mathbf{I}\operatorname{div}\mathbf{u}] - hc_{13}\nabla\rho - he_{31}(\nabla\chi + \nabla\xi) = \mathbf{f} \\ & -\frac{h^5}{720}\bar{c}_{11}\operatorname{div}[(1-\bar{v})\widetilde{\nabla}\boldsymbol{\lambda} + \bar{v}\mathbf{I}\operatorname{div}\boldsymbol{\lambda}] - \frac{h^3}{12}\bar{e}_{31}\nabla\xi + \frac{h^3}{12}c_{44}(\boldsymbol{\lambda} + \nabla\rho) + \frac{h^3}{12}e_{15}\nabla\chi = \mathbf{g} \\ & -\frac{h^3}{12}c_{44}\operatorname{div}(\boldsymbol{\lambda} + \nabla\rho) - \frac{h^3}{12}e_{15}\Delta\chi + hc_{13}\operatorname{div}\mathbf{u} + hc_{33}\rho + he_{33}(\chi+\xi) = a \\ & -\frac{h^3}{12}e_{15}\operatorname{div}(\boldsymbol{\lambda} + \nabla\rho) + \frac{h^3}{12}\varepsilon_{11}\Delta\chi + he_{31}\operatorname{div}\mathbf{u} + he_{33}\rho - h\varepsilon_{33}(\chi+\xi) = -\theta \\ & \frac{h^3}{28}\bar{\varepsilon}_{11}\Delta\xi + he_{31}\operatorname{div}\mathbf{u} + \frac{h^3}{12}\bar{e}_{31}\operatorname{div}\boldsymbol{\lambda} + he_{33}\rho - h\varepsilon_{33}(\chi+\xi) - 5h\bar{\varepsilon}_{33}\xi = -\kappa\end{aligned} \tag{34}$$

At the author's knowledge, this model was not explicitly considered in the literature.

6. Case-study problems

In this section, some case-study problems are introduced in order to perform a comparison between the different piezoelectric plate theories previously reported. To this end, a square piezoelectric plate is considered, whose side is L and whose thickness is h. The x_1 and x_2 axes of the Cartesian frame are parallel to the sides of the plate and the origin O is at one corner of the middle cross-section. The plate is simply supported and electrically grounded along its boundary (Bisegna and Maceri, 1996a). The different problems are characterized by different choices of the external loads, which are applied only on the upper and lower surfaces of the piezoelectric plate (i.e., no volume loads or charges are considered). The problems reported in Section 6.1 and identified by the names $a_f \div d_f$ consider loading conditions able to induce a pure flexural behaviour in the plate. On the other hand, the loading conditions considered in Section 6.2, whose names are $a_m \div d_m$, induce a pure membranal behaviour.

6.1. Flexural problems

(a_f)

$$\begin{aligned}
\mathbf{p}^+ &= q\left(\cos\frac{\pi x_1}{L}\sin\frac{\pi x_2}{L}, \sin\frac{\pi x_1}{L}\cos\frac{\pi x_2}{L}\right)\\
\mathbf{p}^- &= -q\left(\cos\frac{\pi x_1}{L}\sin\frac{\pi x_2}{L}, \sin\frac{\pi x_1}{L}\cos\frac{\pi x_2}{L}\right)\\
p^+ &= p^- = 0\\
\omega^+ &= \omega^- = 0
\end{aligned}\tag{35}$$

(b_f)

$$\begin{aligned}
\mathbf{p}^+ &= q\left(\cos\frac{\pi x_1}{L}\sin\frac{\pi x_2}{L}, -\sin\frac{\pi x_1}{L}\cos\frac{\pi x_2}{L}\right)\\
\mathbf{p}^- &= -q\left(\cos\frac{\pi x_1}{L}\sin\frac{\pi x_2}{L}, -\sin\frac{\pi x_1}{L}\cos\frac{\pi x_2}{L}\right)\\
p^+ &= p^- = 0\\
\omega^+ &= \omega^- = 0
\end{aligned}\tag{36}$$

(c_f)

$$\begin{aligned}
\mathbf{p}^+ &= \mathbf{p}^- = 0\\
p^+ &= \frac{q}{2}\sin\frac{\pi x_1}{L}\sin\frac{\pi x_2}{L}\\
p^- &= \frac{q}{2}\sin\frac{\pi x_1}{L}\sin\frac{\pi x_2}{L}\\
\omega^+ &= \omega^- = 0
\end{aligned}\tag{37}$$

(d_f)

$$\begin{aligned}
\mathbf{p}^+ &= \mathbf{p}^- = 0\\
p^+ &= p^- = 0\\
\omega^+ &= \frac{\omega}{2}\sin\frac{\pi x_1}{L}\sin\frac{\pi x_2}{L}\\
\omega^- &= \frac{\omega}{2}\sin\frac{\pi x_1}{L}\sin\frac{\pi x_2}{L}
\end{aligned}\tag{38}$$

6.2. Membranal problems

(a_{m})

$$\begin{aligned}
\mathbf{p}^{+} &= \frac{q}{2}\left(\cos\frac{\pi x_1}{L}\sin\frac{\pi x_2}{L}, \sin\frac{\pi x_1}{L}\cos\frac{\pi x_2}{L}\right)\\
\mathbf{p}^{-} &= \frac{q}{2}\left(\cos\frac{\pi x_1}{L}\sin\frac{\pi x_2}{L}, \sin\frac{\pi x_1}{L}\cos\frac{\pi x_2}{L}\right)\\
p^{+} &= p^{-} = 0\\
\omega^{+} &= \omega^{-} = 0
\end{aligned} \tag{39}$$

(b_{m})

$$\begin{aligned}
\mathbf{p}^{+} &= \frac{q}{2}\left(\cos\frac{\pi x_1}{L}\sin\frac{\pi x_2}{L}, -\sin\frac{\pi x_1}{L}\cos\frac{\pi x_2}{L}\right)\\
\mathbf{p}^{-} &= \frac{q}{2}\left(\cos\frac{\pi x_1}{L}\sin\frac{\pi x_2}{L}, -\sin\frac{\pi x_1}{L}\cos\frac{\pi x_2}{L}\right)\\
p^{+} &= p^{-} = 0\\
\omega^{+} &= \omega^{-} = 0
\end{aligned} \tag{40}$$

(c_{m})

$$\begin{aligned}
\mathbf{p}^{+} &= \mathbf{p}^{-} = 0\\
p^{+} &= q\sin\frac{\pi x_1}{L}\sin\frac{\pi x_2}{L}\\
p^{-} &= -q\sin\frac{\pi x_1}{L}\sin\frac{\pi x_2}{L}\\
\omega^{+} &= \omega^{-} = 0
\end{aligned} \tag{41}$$

(d_{m})

$$\begin{aligned}
\mathbf{p}^{+} &= \mathbf{p}^{-} = 0\\
p^{+} &= p^{-} = 0\\
\omega^{+} &= \omega\sin\frac{\pi x_1}{L}\sin\frac{\pi x_2}{L}\\
\omega^{-} &= -\omega\sin\frac{\pi x_1}{L}\sin\frac{\pi x_2}{L}
\end{aligned} \tag{42}$$

The three-dimensional analytical solutions of these problems, in the limit of vanishing plate thickness aspect ratio, can be found in closed form as shown by Bisegna and Maceri (1996a).

In order to find out the solutions of these problems according to each of the previously reported piezoelectric plate theories, the following representations are assumed for the unknowns:

$$
\begin{aligned}
w &= w^0 \sin\frac{\pi x_1}{L} \sin\frac{\pi x_2}{L} \\
\alpha &= \alpha^0 \sin\frac{\pi x_1}{L} \sin\frac{\pi x_2}{L} \\
\rho &= \rho^0 \sin\frac{\pi x_1}{L} \sin\frac{\pi x_2}{L} \\
\boldsymbol{\varphi} &= \left(\varphi_1^0 \cos\frac{\pi x_1}{L} \sin\frac{\pi x_2}{L}, \varphi_2^0 \sin\frac{\pi x_1}{L} \cos\frac{\pi x_2}{L}\right) \\
\boldsymbol{\psi} &= \left(\psi_1^0 \cos\frac{\pi x_1}{L} \sin\frac{\pi x_2}{L}, \psi_2^0 \sin\frac{\pi x_1}{L} \cos\frac{\pi x_2}{L}\right) \\
\mathbf{u} &= \left(u_1^0 \cos\frac{\pi x_1}{L} \sin\frac{\pi x_2}{L}, u_2^0 \sin\frac{\pi x_1}{L} \cos\frac{\pi x_2}{L}\right) \\
\boldsymbol{\lambda} &= \left(\lambda_1^0 \cos\frac{\pi x_1}{L} \sin\frac{\pi x_2}{L}, \lambda_2^0 \sin\frac{\pi x_1}{L} \cos\frac{\pi x_2}{L}\right) \\
\eta &= \eta^0 \sin\frac{\pi x_1}{L} \sin\frac{\pi x_2}{L} \\
\upsilon &= \upsilon^0 \sin\frac{\pi x_1}{L} \sin\frac{\pi x_2}{L} \\
\beta &= \beta^0 \sin\frac{\pi x_1}{L} \sin\frac{\pi x_2}{L} \\
\chi &= \chi^0 \sin\frac{\pi x_1}{L} \sin\frac{\pi x_2}{L} \\
\xi &= \xi^0 \sin\frac{\pi x_1}{L} \sin\frac{\pi x_2}{L}
\end{aligned} \tag{43}
$$

It is easy to verify that the boundary conditions are satisfied by these representations of the unknowns. By substituting Eq. (43) into the field equations relevant to each piezoelectric plate theory, linear algebraic systems are obtained for the unknowns w^0, α^0, ρ^0, $\boldsymbol{\varphi}^0$, $\boldsymbol{\psi}^0$, $\mathbf{u}^0$, $\boldsymbol{\lambda}^0$, η^0, υ^0, β^0, χ^0 and ξ^0, whose solutions can be found in closed form.

7. Comparative analysis

The solutions of the flexural problems introduced in Section 6.1 are reported in Tables 3–6. Tables 7–10 show the results relevant to the membranal problems, introduced in Section 6.2.

The solutions supplied by the three-dimensional theory of piezoelectricity, denoted in the following tables by the acronym 3D, are used as a benchmark for the solutions obtained by adopting the different piezoelectric plate theories reported in Sections 4 and 5. In order to make an analytical comparison in the limit of vanishing plate-thickness aspect ratio, only the leading-order terms of the solutions relevant to each theory, expanded as power series with respect to the plate thickness h, are given in the tables. Indeed, only the amplitudes of the relevant quantities, varying according to a sinusoidal law in the x_1, x_2 plane as stated in Eq. (43), are reported.

The comparison is made by taking into account the displacement field (s_1, s_2, s_3), the electric potential ϕ, the total stress field $(T_{11}, T_{12}, T_{13} \| T_{12}, T_{22}, T_{23} \| T_{13}, T_{23}, T_{33})$ and the total electric displacement field (d_1, d_2, d_3).

Table 3
Bending behaviour – loading condition a_f: displacement and electric potential, stress and electric displacement

	$s_1 = s_2$	s_3	ϕ	
3D	$(6qL/\pi^2\delta\hat{c}_{11})\zeta$	$-(6qL/\pi^3\delta^2\hat{c}_{11})$	$-(qL\bar{e}_{31}/\pi\bar{\varepsilon}_{33}\hat{c}_{11})\mathscr{L}_2$	
KL/Q	$(6qL/\pi^2\delta\hat{c}_{11})\zeta$	$-(6qL/\pi^3\delta^2\hat{c}_{11})$	$-(qL\bar{e}_{31}/\pi\bar{\varepsilon}_{33}\hat{c}_{11})\mathscr{L}_2$	
RM/C	$(6qL/\pi^2\delta\hat{c}_{11})\zeta$	$-(6qL/\pi^3\delta^2\hat{c}_{11})$	0	
RM/Q	$(6qL/\pi^2\delta\hat{c}_{11})\zeta$	$-(6qL/\pi^3\delta^2\hat{c}_{11})$	$-(qL\bar{e}_{31}/\pi\bar{\varepsilon}_{33}\hat{c}_{11})\mathscr{L}_2$	
LCW/Q	$(6qL/\pi^2\delta\hat{c}_{11})\zeta$	$-(6qL/\pi^3\delta^2\hat{c}_{11})$	$-(qL\bar{e}_{31}/\pi\bar{\varepsilon}_{33}\hat{c}_{11})\mathscr{L}_2$	
LCW/C	$(6qL/\pi^2\delta\hat{c}_{11})\zeta$	$-(6qL/\pi^3\delta^2\hat{c}_{11})$	$-(qL\bar{e}_{31}/\pi\bar{\varepsilon}_{33}\hat{c}_{11})\mathscr{L}_2$	
	$T_{11} = T_{22}$	T_{12}	$T_{13} = T_{23}$	T_{33}
3D	$-(6q/\pi\delta)(1+\hat{v})\zeta$	$(6q/\pi\delta)(1-\hat{v})\zeta$	$q\mathscr{L}_2$	$4q\pi\delta\mathscr{P}_2$
KL/Q	$-(6q/\pi\delta)(1+\hat{v})\zeta$	$(6q/\pi\delta)(1-\hat{v})\zeta$	$q\mathscr{L}_2$	$4q\pi\delta\mathscr{P}_2$
RM/C	$-(6q/\pi\delta)(1+\hat{v})\zeta$	$(6q/\pi\delta)(1-\hat{v})\zeta$	$q\mathscr{L}_2$	$4q\pi\delta\mathscr{P}_2$
RM/Q	$-(6q/\pi\delta)(1+\hat{v})\zeta$	$(6q/\pi\delta)(1-\hat{v})\zeta$	$q\mathscr{L}_2$	$4q\pi\delta\mathscr{P}_2$
LCW/Q	$-(6q/\pi\delta)(1+\hat{v})\zeta$	$(6q/\pi\delta)(1-\hat{v})\zeta$	$q\mathscr{L}_2$	$4q\pi\delta\mathscr{P}_2$
LCW/C	$-(6q/\pi\delta)(1+\hat{v})\zeta$	$(6q/\pi\delta)(1-\hat{v})\zeta$	$q\mathscr{L}_2$	$4q\pi\delta\mathscr{P}_2$
	$d_1 = d_2$	d_3		
3D	$q(A+B)\mathscr{L}_2$	$4q\pi\delta(A+B)\mathscr{P}_2$		
KL/Q	$qB\mathscr{L}_2$	$4q\pi\delta B\mathscr{P}_2$		
RM/C	0	0		
RM/Q	$qB\mathscr{L}_2$	$4q\pi\delta B\mathscr{P}_2$		
LCW/Q	$q(A+B)\mathscr{L}_2$	$4q\pi\delta(A+B)\mathscr{P}_2$		
LCW/C	$q(A+B)\mathscr{L}_2$	$4q\pi\delta(A+B)\mathscr{P}_2$		

Table 4
Bending behaviour – loading condition b_f: displacement and electric potential, stress and electric displacement

	$s_1 = -s_2$	s_3	ϕ	
3D	$(q\delta L/c_{44})\zeta$	0	0	
KL/Q	0	0	0	
RM/C	$(q\delta L/c_{44})\zeta$	0	0	
RM/Q	$(q\delta L/c_{44})\zeta$	0	0	
LCW/Q	$(q\delta L/c_{44})\zeta$	0	0	
LCW/C	$(q\delta L/c_{44})\zeta$	0	0	
	$T_{11} = -T_{22}$	T_{12}	$T_{13} = -T_{23}$	T_{33}
3D	$-(2q\pi\delta c_{66}/c_{44})\zeta$	0	q	0
KL/Q	0	0	q	0
RM/C	$-(2q\pi\delta c_{66}/c_{44})\zeta$	0	q	0
RM/Q	$-(2q\pi\delta c_{66}/c_{44})\zeta$	0	q	0
LCW/Q	$-(2q\pi\delta c_{66}/c_{44})\zeta$	0	q	0
LCW/C	$-(2q\pi\delta c_{66}/c_{44})\zeta$	0	q	0
	$d_1 = -d_2$	d_3		
3D	qe_{15}/c_{44}	0		
KL/Q	0	0		
RM/C	qe_{15}/c_{44}	0		
RM/Q	qe_{15}/c_{44}	0		
LCW/Q	qe_{15}/c_{44}	0		
LCW/C	qe_{15}/c_{44}	0		

Table 5
Bending behaviour – loading condition c_f: displacement and electric potential, stress and electric displacement

	$s_1 = s_2$	s_3	ϕ	
3D	$-(3qL/\pi^3\delta^2\hat{c}_{11})\zeta$	$(3qL/\pi^4\delta^3\hat{c}_{11})$	$(qLe_{15}/2\pi^2\delta\overline{\varepsilon}_{11}c_{44}) + (qL\overline{e}_{31}/2\pi^2\delta\overline{\varepsilon}_{33}\hat{c}_{11})\mathscr{L}_2$	
KL/Q	$-(3qL/\pi^3\delta^2\hat{c}_{11})\zeta$	$(3qL/\pi^4\delta^3\hat{c}_{11})$	$(qL\overline{e}_{31}/2\pi^2\delta\overline{\varepsilon}_{33}\hat{c}_{11})\mathscr{L}_2$	
RM/C	$-(3qL/\pi^3\delta^2\hat{c}_{11})\zeta$	$(3qL/\pi^4\delta^3\hat{c}_{11})$	$(qLe_{15}/2\pi^2\delta\overline{\varepsilon}_{11}c_{44})$	
RM/Q	$-(3qL/\pi^3\delta^2\hat{c}_{11})\zeta$	$(3qL/\pi^4\delta^3\hat{c}_{11})$	$(qLe_{15}/2\pi^2\delta\overline{\varepsilon}_{11}c_{44}) + (qL\overline{e}_{31}/2\pi^2\delta\overline{\varepsilon}_{33}\hat{c}_{11})\mathscr{L}_2$	
LCW/Q	$-(3qL/\pi^3\delta^2\hat{c}_{11})\zeta$	$(3qL/\pi^4\delta^3\hat{c}_{11})$	$(qLe_{15}/2\pi^2\delta\overline{\varepsilon}_{11}c_{44}) + (qL\overline{e}_{31}/2\pi^2\delta\overline{\varepsilon}_{33}\hat{c}_{11})\mathscr{L}_2$	
LCW/C	$-(3qL/\pi^3\delta^2\hat{c}_{11})\zeta$	$(3qL/\pi^4\delta^3\hat{c}_{11})$	$(qLe_{15}/2\pi^2\delta\overline{\varepsilon}_{11}c_{44}) + (qL\overline{e}_{31}/2\pi^2\delta\overline{\varepsilon}_{33}\hat{c}_{11})\mathscr{L}_2$	
	$T_{11} = T_{22}$	T_{12}	$T_{13} = T_{23}$	T_{33}
3D	$(3q/\pi^2\delta^2)(1+\hat{v})\zeta$	$-(3q/\pi^2\delta^2)(1-\hat{v})\zeta$	$-(3q/\pi\delta)\mathscr{P}_1$	$-(q/2)\mathscr{P}_3$
KL/Q	$(3q/\pi^2\delta^2)(1+\hat{v})\zeta$	$-(3q/\pi^2\delta^2)(1-\hat{v})\zeta$	$-(3q/\pi\delta)\mathscr{P}_1$	$-(q/2)\mathscr{P}_3$
RM/C	$(3q/\pi^2\delta^2)(1+\hat{v})\zeta$	$-(3q/\pi^2\delta^2)(1-\hat{v})\zeta$	$-(3q/\pi\delta)\mathscr{P}_1$	$-(q/2)\mathscr{P}_3$
RM/Q	$(3q/\pi^2\delta^2)(1+\hat{v})\zeta$	$-(3q/\pi^2\delta^2)(1-\hat{v})\zeta$	$-(3q/\pi\delta)\mathscr{P}_1$	$-(q/2)\mathscr{P}_3$
LCW/Q	$(3q/\pi^2\delta^2)(1+\hat{v})\zeta$	$-(3q/\pi^2\delta^2)(1-\hat{v})\zeta$	$-(3q/\pi\delta)\mathscr{P}_1$	$-(q/2)\mathscr{P}_3$
LCW/C	$(3q/\pi^2\delta^2)(1+\hat{v})\zeta$	$-(3q/\pi^2\delta^2)(1-\hat{v})\zeta$	$-(3q/\pi\delta)\mathscr{P}_1$	$-(q/2)\mathscr{P}_3$
	$d_1 = d_2$	d_3		
3D	$-(qL/2\pi h)(A+B)\mathscr{L}_2$	$-2q(A+B)\mathscr{P}_2$		
KL/Q	$-(qL/2\pi h)B\mathscr{L}_2$	$-2qB\mathscr{P}_2$		
RM/C	0	0		
RM/Q	$-(qL/2\pi h)B\mathscr{L}_2$	$-2qB\mathscr{P}_2$		
LCW/Q	$-(qL/2\pi h)(A+B)\mathscr{L}_2$	$-2q(A+B)\mathscr{P}_2$		
LCW/C	$-(qL/2\pi h)(A+B)\mathscr{L}_2$	$-2q(A+B)\mathscr{P}_2$		

For brevity, the following positions are understood:

$$\delta = h/L \qquad A = \frac{e_{15}}{c_{44}} \qquad B = \frac{\overline{\varepsilon}_{11}\overline{e}_{31}}{\hat{c}_{11}\overline{\varepsilon}_{33}} \tag{44}$$

7.1. Flexural theories

The Kirchhoff–Love type theory KL/Q is based on the assumption of vanishing transversal strain and so it fails in evaluating the terms of the three-dimensional solution related to the e_{15} piezoelectric material constant, due to the coupling between the in-plane electric field $\mathbf{e}$ and the transversal strain γ; in particular, in the case of the loading condition c_f this theory evaluates only the quadratic term relevant to the three-dimensional solution for the electric potential ϕ, and gives zero values for all the unknowns in the case of the flexural loading condition b_f.

The Reissner–Mindlin type theory RM/L employs a first-order representation of the displacement and electric field: therefore, it is able to evaluate the mean value of the higher-order terms contained in some of the solutions supplied by the three-dimensional theory. In particular, in the case of the loading condition c_f this model evaluates the mean-value in the thickness of the electric potential ϕ and finds null values for the electric displacement in the case of the loading conditions a_f and c_f.

The Reissner–Mindlin type theory with quadratic electric potential RM/Q gives better results because it considers a non-vanishing transversal strain γ together with a quadratic electric potential in the thickness. In particular, in the loading condition c_f it is able to evaluate the correct value of the electric potential ϕ.

Table 6
Bending behaviour – loading condition d_f: displacement and electric potential, stress and electric displacement

	$s_1 = s_2$	s_3	ϕ	
3D	$(\omega L\bar{e}_{31}/2\pi\bar{\varepsilon}_{33}\hat{c}_{11})\zeta$	$-(\omega L\bar{e}_{31}/2\pi^2\delta\bar{\varepsilon}_{33}\hat{c}_{11}) - (\omega Le_{15}/2\pi^2\delta\bar{\varepsilon}_{11}c_{44})$	$(\omega L/2\pi^2\delta\bar{\varepsilon}_{11})$	
KL/Q	$(\omega L\bar{e}_{31}/2\pi\bar{\varepsilon}_{33}\hat{c}_{11})\zeta$	$-(\omega L\bar{e}_{31}/2\pi^2\delta\bar{\varepsilon}_{33}\hat{c}_{11})$	$(\omega L/2\pi^2\delta\bar{\varepsilon}_{11})$	
RM/C	0	$-(\omega Le_{15}/2\pi^2\delta\bar{\varepsilon}_{11}c_{44})$	$(\omega L/2\pi^2\delta\bar{\varepsilon}_{11})$	
RM/Q	$(\omega L\bar{e}_{31}/2\pi\bar{\varepsilon}_{33}\hat{c}_{11})\zeta$	$-(\omega L\bar{e}_{31}/2\pi^2\delta\bar{\varepsilon}_{33}\hat{c}_{11}) - (\omega Le_{15}/2\pi^2\delta\bar{\varepsilon}_{11}c_{44})$	$(\omega L/2\pi^2\delta\bar{\varepsilon}_{11})$	
LCW/Q	$(\omega L\bar{e}_{31}/2\pi\bar{\varepsilon}_{33}\hat{c}_{11})\zeta$	$-(\omega L\bar{e}_{31}/2\pi^2\delta\bar{\varepsilon}_{33}\hat{c}_{11}) - (\omega Le_{15}/2\pi^2\delta\bar{\varepsilon}_{11}c_{44})$	$(\omega L/2\pi^2\delta\bar{\varepsilon}_{11})$	
LCW/C	$(\omega L\bar{e}_{31}/2\pi\bar{\varepsilon}_{33}\hat{c}_{11})\zeta$	$-(\omega L\bar{e}_{31}/2\pi^2\delta\bar{\varepsilon}_{33}\hat{c}_{11}) - (\omega Le_{15}/2\pi^2\delta\bar{\varepsilon}_{11}c_{44})$	$(\omega L/2\pi^2\delta\bar{\varepsilon}_{11})$	
	$T_{11} = T_{22}$	T_{12}	$T_{13} = T_{23}$	T_{33}
3D	$(\omega c_{66}\bar{e}_{31}/\bar{\varepsilon}_{33}\hat{c}_{11})\zeta$	$(\omega c_{66}\bar{e}_{31}/\bar{\varepsilon}_{33}\hat{c}_{11})\zeta$	$(\pi^3\delta^3\omega/12)((\hat{e}_{33}/\bar{\varepsilon}_{33}) - (A+B)(\bar{c}_{11}/\bar{\varepsilon}_{33}))\mathscr{P}_4$	$(\pi^4\delta^4\omega/3)((\hat{e}_{33}/\bar{\varepsilon}_{33}) - (A+B)(\bar{c}_{11}/\bar{\varepsilon}_{33}))\mathscr{P}_6$
KL/Q	$(\omega c_{66}\bar{e}_{31}/\bar{\varepsilon}_{33}\hat{c}_{11})\zeta$	$(\omega c_{66}\bar{e}_{31}/\bar{\varepsilon}_{33}\hat{c}_{11})\zeta$	0	0
RM/C	0	0	0	0
RM/Q	$(\omega c_{66}\bar{e}_{31}/\bar{\varepsilon}_{33}\hat{c}_{11})\zeta$	$(\omega c_{66}\bar{e}_{31}/\bar{\varepsilon}_{33}\hat{c}_{11})\zeta$	0	0
LCW/Q	$(\omega c_{66}\bar{e}_{31}/\bar{\varepsilon}_{33}\hat{c}_{11})\zeta$	$(\omega c_{66}\bar{e}_{31}/\bar{\varepsilon}_{33}\hat{c}_{11})\zeta$	$(\pi^3\delta^3\omega/12)((\hat{e}_{33}/\bar{\varepsilon}_{33}) - A(\bar{c}_{11}/\bar{\varepsilon}_{33}))\mathscr{P}_4$	$(\pi^4\delta^4\omega/3)((\hat{e}_{33}/\bar{\varepsilon}_{33}) - A(\bar{c}_{11}/\bar{\varepsilon}_{33}))\mathscr{P}_6$
LCW/C	$(\omega c_{66}\bar{e}_{31}/\bar{\varepsilon}_{33}\hat{c}_{11})\zeta$	$(\omega c_{66}\bar{e}_{31}/\bar{\varepsilon}_{33}\hat{c}_{11})\zeta$	$(\pi^3\delta^3\omega/12)((\hat{e}_{33}/\bar{\varepsilon}_{33}) - (A+B)(\bar{c}_{11}/\bar{\varepsilon}_{33}))\mathscr{P}_4$	$(\pi^4\delta^4\omega/3)((\hat{e}_{33}/\bar{\varepsilon}_{33}) - (A+B)(\bar{c}_{11}/\bar{\varepsilon}_{33}))\mathscr{P}_6$
	$d_1 = d_2$	d_3		
3D	$-(\omega/2\pi\delta)$	$-\omega\zeta$		
KL/Q	$-(\omega/2\pi\delta)$	$-\omega\zeta$		
RM/C	$-(\omega/2\pi\delta)$	$-\omega\zeta$		
RM/Q	$-(\omega/2\pi\delta)$	$-\omega\zeta$		
LCW/Q	$-(\omega/2\pi\delta)$	$-\omega\zeta$		
LCW/C	$-(\omega/2\pi\delta)$	$-\omega\zeta$		

The theory LCW/Q, based on a Lo–Christensen–Wu type representation of the displacement field, considers a quadratic transversal strain and so it is able to model the coupling between the quadratic in-plane electric field **e** and the quadratic transversal strain γ; it provides the correct values of the unknown fields for almost all the flexural loading conditions considered; in particular it is able to evaluate the correct values of the electric displacement in the case of loading conditions a_f and c_f. It fails only in evaluating the correct values of the transversal stresses $T_{13} = T_{23}$ and T_{33} in the case of the loading condition d_f.

It is emphasized that the previously discussed theories estimate a null value for these unknowns. In order to exactly estimate also these unknowns it is necessary to consider higher-order terms in the representation of the electric potential field; the theory LCW/B, which considers a biquadratic variation of the electric potential along the plate thickness, is able to correctly estimate the leading order terms of the unknown fields for all the loading conditions considered.

7.2. Membranal theories

The classical theory C/L is based on the assumption of null transversal elongation ε and so it is not able to estimate the value of s_3 in any case and provides a null solution in the case of the loading condition c_m and null values for the transversal stresses in the case of the loading condition d_m.

The theory CT/L, based on a constant value of the transversal elongation ε and a null value of the transversal strain γ, fails to correctly predict the values of the electric displacement in the case of the loading conditions a_m and b_m and the values of the transversal stresses in the case of the loading conditions c_m and d_m.

Table 7
Membrane behaviour – loading condition a_m: displacement and electric potential, stress and electric displacement

	$s_1 = s_2$	s_3	ϕ	
3D	$(qL/2\pi^2\delta\hat{c}_{11})$	$(qL\hat{\varepsilon}_{33}/\pi\bar{\varepsilon}_{33}\hat{c}_{11})\zeta$	$-(qL\bar{e}_{31}/\pi\bar{\varepsilon}_{33}\hat{c}_{11})\zeta$	
C/L	$(qL/2\pi^2\delta\hat{c}_{11})$	0	$-(qL\bar{e}_{31}/\pi\bar{\varepsilon}_{33}\hat{c}_{11})\zeta$	
CT/L	$(qL/2\pi^2\delta\hat{c}_{11})$	$(qL\hat{\varepsilon}_{33}/\pi\bar{\varepsilon}_{33}\hat{c}_{11})\zeta$	$-(qL\bar{e}_{31}/\pi\bar{\varepsilon}_{33}\hat{c}_{11})\zeta$	
CTS/L	$(qL/2\pi^2\delta\hat{c}_{11})$	$(qL\hat{\varepsilon}_{33}/\pi\bar{\varepsilon}_{33}\hat{c}_{11})\zeta$	$-(qL\bar{e}_{31}/\pi\bar{\varepsilon}_{33}\hat{c}_{11})\zeta$	
CTS/C	$(qL/2\pi^2\delta\hat{c}_{11})$	$(qL\hat{\varepsilon}_{33}/\pi\bar{\varepsilon}_{33}\hat{c}_{11})\zeta$	$-(qL\bar{e}_{31}/\pi\bar{\varepsilon}_{33}\hat{c}_{11})\zeta$	
	$T_{11} = T_{22}$	T_{12}	$T_{13} = T_{23}$	T_{33}
3D	$-(q/2\pi\delta)(1+\hat{v})$	$(q/2\pi\delta)(1-\hat{v})$	$q\zeta$	$q\pi\delta\mathscr{P}_1$
C/L	$-(q/2\pi\delta)(1+\hat{v})$	$(q/2\pi\delta)(1-\hat{v})$	$q\zeta$	$q\pi\delta\mathscr{P}_1$
CT/L	$-(q/2\pi\delta)(1+\hat{v})$	$(q/2\pi\delta)(1-\hat{v})$	$q\zeta$	$q\pi\delta\mathscr{P}_1$
CTS/L	$-(q/2\pi\delta)(1+\hat{v})$	$(q/2\pi\delta)(1-\hat{v})$	$q\zeta$	$q\pi\delta\mathscr{P}_1$
CTS/C	$-(q/2\pi\delta)(1+\hat{v})$	$(q/2\pi\delta)(1-\hat{v})$	$q\zeta$	$q\pi\delta\mathscr{P}_1$
	$d_1 = d_2$	d_3		
3D	$q(A+B)\zeta$	$q\pi\delta(A+B)\mathscr{P}_1$		
C/L	$qB\zeta$	$q\pi\delta B\mathscr{P}_1$		
CT/L	$qB\zeta$	$q\pi\delta B\mathscr{P}_1$		
CTS/L	$q(A+B)\zeta$	$q\pi\delta(A+B)\mathscr{P}_1$		
CTS/C	$q(A+B)\zeta$	$q\pi\delta(A+B)\mathscr{P}_1$		

Table 8
Membrane behaviour – loading condition b_m: displacement and electric potential, stress and electric displacement

	$s_1 = -s_2$	s_3	ϕ	
3D	$(qL/2\pi^2\delta c_{66})$	0	0	
C/L	$(qL/2\pi^2\delta c_{66})$	0	0	
CT/L	$(qL/2\pi^2\delta c_{66})$	0	0	
CTS/L	$(qL/2\pi^2\delta c_{66})$	0	0	
CTS/C	$(qL/2\pi^2\delta c_{66})$	0	0	
	$T_{11} = -T_{22}$	T_{12}	$T_{13} = -T_{23}$	T_{33}
3D	$-(q/\pi\delta)$	0	$q\zeta$	0
C/L	$-(q/\pi\delta)$	0	$q\zeta$	0
CT/L	$-(q/\pi\delta)$	0	$q\zeta$	0
CTS/L	$-(q/\pi\delta)$	0	$q\zeta$	0
CTS/C	$-(q/\pi\delta)$	0	$q\zeta$	0
	$d_1 = -d_2$	d_3		
3D	$(qe_{15}/c_{44})\zeta$	0		
C/L	0	0		
CT/L	0	0		
CTS/L	$(qe_{15}/c_{44})\zeta$	0		
CTS/C	$(qe_{15}/c_{44})\zeta$	0		

A better, but not completely satisfactory, estimation of the transversal stresses in the case of the loading conditions c_m and d_m is achieved with the theory CTS/L, which considers a linear transversal strain along the thickness of the plate and so it is able to model the coupling between the linear in-plane electric field $\mathbf{e}$ and the linear transversal strain $\boldsymbol{\gamma}$. This theory evaluates correctly the values of the electric displacement in the case of the loading conditions a_m and b_m.

Table 9
Membrane behaviour – loading condition c_m: displacement and electric potential, stress and electric displacement

	$s_1 = s_2$	s_3	ϕ
3D	$(qL\hat{\varepsilon}_{33}/2\pi\bar{\varepsilon}_{33}\hat{c}_{11})$	$(qL\tilde{\varepsilon}_{33}/\bar{\varepsilon}_{33}\hat{c}_{11})\delta\zeta$	$(qL\hat{e}_{33}/\bar{\varepsilon}_{33}\hat{c}_{11})\delta\zeta$
C/L	0	0	0
CT/L	$(qL\hat{\varepsilon}_{33}/2\pi\bar{\varepsilon}_{33}\hat{c}_{11})$	$(qL\tilde{\varepsilon}_{33}/\bar{\varepsilon}_{33}\hat{c}_{11})\delta\zeta$	$(qL\hat{e}_{33}/\bar{\varepsilon}_{33}\hat{c}_{11})\delta\zeta$
CTS/L	$(qL\hat{\varepsilon}_{33}/2\pi\bar{\varepsilon}_{33}\hat{c}_{11})$	$(qL\tilde{\varepsilon}_{33}/\bar{\varepsilon}_{33}\hat{c}_{11})\delta\zeta$	$(qL\hat{e}_{33}/\bar{\varepsilon}_{33}\hat{c}_{11})\delta\zeta$
CTS/C	$(qL\hat{\varepsilon}_{33}/2\pi\bar{\varepsilon}_{33}\hat{c}_{11})$	$(qL\tilde{\varepsilon}_{33}/\bar{\varepsilon}_{33}\hat{c}_{11})\delta\zeta$	$(qL\hat{e}_{33}/\bar{\varepsilon}_{33}\hat{c}_{11})\delta\zeta$

	$T_{11} = T_{22}$	T_{12}	$T_{13} = T_{23}$	T_{33}
3D	$(q\hat{\varepsilon}_{33}/2\bar{\varepsilon}_{33})(1-\hat{\nu})$	$(q\hat{\varepsilon}_{33}/\bar{\varepsilon}_{33})(1-\hat{\nu})$	$-(q\pi^3\delta^3/3)(1/\bar{\varepsilon}_{33})(\tilde{\varepsilon}_{33}+(A+B)\hat{e}_{33})\mathscr{P}_2$	q
C/L	0	0	0	q
CT/L	$(q\hat{\varepsilon}_{33}/2\bar{\varepsilon}_{33})(1-\hat{\nu})$	$(q\hat{\varepsilon}_{33}/\bar{\varepsilon}_{33})(1-\hat{\nu})$	$-(q\pi^3\delta^3/3)(\tilde{\varepsilon}_{33}/\bar{\varepsilon}_{33})\mathscr{P}_2$	q
CTS/L	$(q\hat{\varepsilon}_{33}/2\bar{\varepsilon}_{33})(1-\hat{\nu})$	$(q\hat{\varepsilon}_{33}/\bar{\varepsilon}_{33})(1-\hat{\nu})$	$-(q\pi^3\delta^3/3)(1/\bar{\varepsilon}_{33})(\tilde{\varepsilon}_{33}+A\hat{e}_{33})\mathscr{P}_2$	q
CTS/C	$(q\hat{\varepsilon}_{33}/2\bar{\varepsilon}_{33})(1-\hat{\nu})$	$(q\hat{\varepsilon}_{33}/\bar{\varepsilon}_{33})(1-\hat{\nu})$	$-(q\pi^3\delta^3/3)(1/\bar{\varepsilon}_{33})(\tilde{\varepsilon}_{33}+(A+B)\hat{e}_{33})\mathscr{P}_2$	q

	$d_1 = d_2$	d_3
3D	$-q\pi\delta(\bar{\varepsilon}_{11}\hat{e}_{33}/\bar{\varepsilon}_{33}\hat{c}_{11})\zeta$	$-q\pi^2\delta^2(\bar{\varepsilon}_{11}\hat{e}_{33}/\bar{\varepsilon}_{33}\hat{c}_{11})\mathscr{P}_1$
C/L	0	0
CT/L	$-q\pi\delta(\bar{\varepsilon}_{11}\hat{e}_{33}/\bar{\varepsilon}_{33}\hat{c}_{11})\zeta$	$-q\pi^2\delta^2(\bar{\varepsilon}_{11}\hat{e}_{33}/\bar{\varepsilon}_{33}\hat{c}_{11})\mathscr{P}_1$
CTS/L	$-q\pi\delta(\bar{\varepsilon}_{11}\hat{e}_{33}/\bar{\varepsilon}_{33}\hat{c}_{11})\zeta$	$-q\pi^2\delta^2(\bar{\varepsilon}_{11}\hat{e}_{33}/\bar{\varepsilon}_{33}\hat{c}_{11})\mathscr{P}_1$
CTS/C	$-q\pi\delta(\bar{\varepsilon}_{11}\hat{e}_{33}/\bar{\varepsilon}_{33}\hat{c}_{11})\zeta$	$-q\pi^2\delta^2(\bar{\varepsilon}_{11}\hat{e}_{33}/\bar{\varepsilon}_{33}\hat{c}_{11})\mathscr{P}_1$

Table 10
Membrane behaviour – loading condition d_m: displacement and electric potential, stress and electric displacement

	$s_1 = s_2$	s_3	ϕ
3D	$\omega L\bar{e}_{31}/2\pi\bar{\varepsilon}_{33}\hat{c}_{11}$	$-(\omega L\hat{e}_{33}/\bar{\varepsilon}_{33}\hat{c}_{11})\delta\zeta$	$(\omega L\bar{c}_{11}/\bar{\varepsilon}_{33}\hat{c}_{11})\delta\zeta$
C/L	$\omega L\bar{e}_{31}/2\pi\bar{\varepsilon}_{33}\hat{c}_{11}$	0	$(\omega L\bar{c}_{11}/\bar{\varepsilon}_{33}\hat{c}_{11})\delta\zeta$
CT/L	$\omega L\bar{e}_{31}/2\pi\bar{\varepsilon}_{33}\hat{c}_{11}$	$-(\omega L\hat{e}_{33}/\bar{\varepsilon}_{33}\hat{c}_{11})\delta\zeta$	$(\omega L\bar{c}_{11}/\bar{\varepsilon}_{33}\hat{c}_{11})\delta\zeta$
CTS/L	$\omega L\bar{e}_{31}/2\pi\bar{\varepsilon}_{33}\hat{c}_{11}$	$-(\omega L\hat{e}_{33}/\bar{\varepsilon}_{33}\hat{c}_{11})\delta\zeta$	$(\omega L\bar{c}_{11}/\bar{\varepsilon}_{33}\hat{c}_{11})\delta\zeta$
CTS/C	$\omega L\bar{e}_{31}/2\pi\bar{\varepsilon}_{33}\hat{c}_{11}$	$-(\omega L\hat{e}_{33}/\bar{\varepsilon}_{33}\hat{c}_{11})\delta\zeta$	$(\omega L\bar{c}_{11}/\bar{\varepsilon}_{33}\hat{c}_{11})\delta\zeta$

	$T_{11} = T_{22}$	T_{12}	$T_{13} = T_{23}$	T_{33}
3D	$\omega c_{66}\bar{e}_{31}/\bar{\varepsilon}_{33}\hat{c}_{11}$	$\omega c_{66}\bar{e}_{31}/\bar{\varepsilon}_{33}\hat{c}_{11}$	$(\omega\pi^3\delta^3/3)((\hat{e}_{33}/\bar{\varepsilon}_{33})-(\bar{c}_{11}/\bar{\varepsilon}_{33})(A+B))\mathscr{P}_2$	$(\omega\pi^4\delta^4/6)((\hat{e}_{33}/\bar{\varepsilon}_{33})-(\bar{c}_{11}/\bar{\varepsilon}_{33})(A+B))\mathscr{P}_5$
C/L	$\omega c_{66}\bar{e}_{31}/\bar{\varepsilon}_{33}\hat{c}_{11}$	$\omega c_{66}\bar{e}_{31}/\bar{\varepsilon}_{33}\hat{c}_{11}$	0	0
CT/L	$\omega c_{66}\bar{e}_{31}/\bar{\varepsilon}_{33}\hat{c}_{11}$	$\omega c_{66}\bar{e}_{31}/\bar{\varepsilon}_{33}\hat{c}_{11}$	$(\omega\pi^3\delta^3/3)(\hat{e}_{33}/\bar{\varepsilon}_{33})\mathscr{P}_2$	$(\omega\pi^4\delta^4/6)(\hat{e}_{33}/\bar{\varepsilon}_{33})\mathscr{P}_5$
CTS/L	$\omega c_{66}\bar{e}_{31}/\bar{\varepsilon}_{33}\hat{c}_{11}$	$\omega c_{66}\bar{e}_{31}/\bar{\varepsilon}_{33}\hat{c}_{11}$	$(\omega\pi^3\delta^3/3)((\hat{e}_{33}/\bar{\varepsilon}_{33})-(\bar{c}_{11}/\bar{\varepsilon}_{33})A)\mathscr{P}_2$	$(\omega\pi^4\delta^4/6)((\hat{e}_{33}/\bar{\varepsilon}_{33})-(\bar{c}_{11}/\bar{\varepsilon}_{33})A)\mathscr{P}_5$
CTS/C	$\omega c_{66}\bar{e}_{31}/\bar{\varepsilon}_{33}\hat{c}_{11}$	$\omega c_{66}\bar{e}_{31}/\bar{\varepsilon}_{33}\hat{c}_{11}$	$(\omega\pi^3\delta^3/3)((\hat{e}_{33}/\bar{\varepsilon}_{33})-(\bar{c}_{11}/\bar{\varepsilon}_{33})(A+B))\mathscr{P}_2$	$(\omega\pi^4\delta^4/6)((\hat{e}_{33}/\bar{\varepsilon}_{33})-(\bar{c}_{11}/\bar{\varepsilon}_{33})(A+B))\mathscr{P}_5$

	$d_1 = d_2$	d_3
3D	$-(\omega\pi\bar{\varepsilon}_{11}\bar{c}_{11}\delta/\bar{\varepsilon}_{33}\hat{c}_{11})\zeta$	$-\omega$
C/L	$-(\omega\pi\bar{\varepsilon}_{11}\bar{c}_{11}\delta/\bar{\varepsilon}_{33}\hat{c}_{11})\zeta$	$-\omega$
CT/L	$-(\omega\pi\bar{\varepsilon}_{11}\bar{c}_{11}\delta/\bar{\varepsilon}_{33}\hat{c}_{11})\zeta$	$-\omega$
CTS/L	$-(\omega\pi\bar{\varepsilon}_{11}\bar{c}_{11}\delta/\bar{\varepsilon}_{33}\hat{c}_{11})\zeta$	$-\omega$
CTS/C	$-(\omega\pi\bar{\varepsilon}_{11}\bar{c}_{11}\delta/\bar{\varepsilon}_{33}\hat{c}_{11})\zeta$	$-\omega$

In order to correctly evaluate also the transversal stresses in the loading cases c_m and d_m it is necessary to consider a more refined representation of the electric potential, which was chosen linear along the plate thickness by the previous theories. The theory CTS/C, which considers the same displacement field adopted in the theory CTS/L together with a cubic variation of the electric potential, turns out to be able to correctly predict the leading order terms of the unknown fields for all the loading conditions considered.

In conclusion, as discussed in the introduction, the theories LCW/B for the flexural behaviour and CTS/C for the membranal behaviour represent the optimal choices of the intermediate-order theories to be adopted, e.g., in the analysis of a thick plate, modeled as a stack of thin layers. In fact, such theories are the lowest-order ones able to exactly reproduce all the unknown fields supplied by the three-dimensional solutions, in the limit of plate thickness aspect ratio approaching zero.

8. Conclusions

A classification and a critical comparison of some existing models for the bending and stretching behaviour of piezoelectric plates were presented. These models were also rationally derived from the three-dimensional theory of piezoelectricity, and a consistent treatment of the stress and electric-displacement relaxation conditions was proposed.

In order to perform the comparison, case-study problems of a simply supported square piezoelectric plate, grounded along its lateral boundary and loaded by different sinusoidal mechanical or electric loads, were considered. The analytical solutions supplied by each of the models under evaluation were compared with the solutions obtained in the framework of the three-dimensional Voigt theory of piezoelectricity. It turned out that, by taking into account only the first few terms in the expansions of the unknown fields, only the displacement and electric-potential fields may be accurately computed. In order to obtain accurate estimates of the stress and electric-displacement fields it was necessary to take into account higher-order terms, up to the fourth-order ones in the electric potential representation.

Further research will be concerned with the development of layerwise refined modelizations of piezoelectric laminates. Such modelizations, obtained by a suitable assembly of the potential-energy functionals of each single layer, should take into account different interface laws describing the contact between adjacent layer, ranging from perfect rigid bonding to elastic bonding, to partial debonding.

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